SPIRAL MINIMAL SURFACES*

BY

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1. **Introduction.** The equations of any minimal surface S may be written†

$$
(1)\qquad
\begin{aligned}
x &= \frac{1}{2}\int (1-u^2)F(u)\,du + \frac{1}{2}\int (1-v^2)\Phi(v)\,dv = U_1 + V_1\\
y &= \frac{i}{2}\int (1+u^2)F(u)\,du - \frac{i}{2}\int (1+v^2)\Phi(v)\,dv = U_2 + V_2\\
z &= \int uF(u)\,du + \int v\Phi(v)\,dv = U_3 + V_3.
\end{aligned}
$$

If the surface is real F and Φ are conjugate functions, and for a real point with a real tangent plane u and v have conjugate values.‡ The direction cosines of the normal to S are

$$
(2)\qquad X = \frac{u+v}{uv+1},\qquad Y = \frac{i(v-u)}{uv+1},\qquad Z = \frac{uv-1}{uv+1}.
$$

The linear element ds is given by

$$ds^2 = (uv+1)^2 F(u)\Phi(v)\,du\,dv.$$

The differential equations of the lines of curvature and of the asymptotic lines of S are respectively

$$F(u)\,du^2 - \Phi(v)\,dv^2 = 0,\qquad F(u)\,du^2 + \Phi(v)\,dv^2 = 0.$$

The curves of any surface along each of which the tangent plane to the surface makes a constant angle with a fixed direction are called the *Minding parallels* of the surface with reference to that direction; the curves of the surface along each of which the normal is parallel to a plane containing the fixed direction are called the *Minding meridians*.§ We shall consider Minding

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† Darboux, *Théorie des surfaces*, vol. 1, 2d ed., p. 340. These equations are due to Enneper (1864) and Weierstrass (1866).

‡ Darboux, l. c., p. 343.

§ These curves were first studied by Minding, Journal für Mathematik, vol. 44 (1852), p. 66. Darboux, l. c., p. 367.

parallels and meridians of minimal surfaces with reference to the Z axis. If in (1) we set $u = re^{\phi i}$ and $v = re^{-\phi i}$, where for real points of a real surface r is real and positive and ϕ is real, it is evident from (2) that the curves (r), that is r constant, are the Minding parallels and the curves (ϕ) the Minding meridians. In the study of spiral minimal surfaces, the subject of this paper, the curves (r) play a part analogous to that taken by the curves (ϕ) in the discussion of minimal surfaces applicable to surfaces of revolution.

If F and Φ in (1) are replaced by $e^{\alpha i} F$ and $e^{-\alpha i}\Phi$ respectively, where α is a real constant, we obtain the equations of a family of minimal surfaces called *associate* to S.* Corresponding points of associate minimal surfaces are points given by the same parameter values u, v. Schwarz proved that the locus of points on a family of associate surfaces corresponding to given values of u, v is an ellipse whose center is the origin. These ellipses we call *Schwarz ellipses*. We have found the equations of the locus L of the vertices of the Schwarz ellipses for a given minimal surface with equations (1) to be†

$$
(3)\qquad \begin{gathered}
x = U_1\sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}} + V_1\sqrt[4]{\frac{\Sigma U_1^2}{\Sigma V_1^2}}, \qquad y = U_2\sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}} + V_2\sqrt[4]{\frac{\Sigma U_1^2}{\Sigma V_1^2}}, \\
z = U_3\sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}} + V_3\sqrt[4]{\frac{\Sigma U_1^2}{\Sigma V_1^2}}.
\end{gathered}
$$

In (3) the first radical of each second member has the same determination; the second is the reciprocal and conjugate of the first for real points of a real surface S. The four vertices of each ellipse are given by the four determinations of the first radical. Evidently L consists generally of four nappes symmetrical in pairs with respect to the origin.

A *spiral surface* is most simply defined as follows:‡ A spiral surface is the locus of the different positions of any curve which is rotated about an axis and at the same time subjected to a homothetic transformation with respect to a point of the axis in such a way that the tangent to the locus described by any point of the curve makes a constant angle with the axis. The spiral surfaces include surfaces of revolution but we shall hereafter in speaking of a spiral surface suppose it not to be a surface of revolution. Evidently the locus of each point of the curve generating a spiral surface lies on a right cir-

* Associate minimal surfaces were first systematically considered by H. A. Schwarz, *Miscellen aus dem Gebiete der Minimalflächen*, Journal für Mathematik, vol. 80 (1875), p. 286, see also Darboux, l. c., p. 379.

† American Journal of Mathematics, vol. 40 (1918), p. 87.

‡ Eisenhart, *Differential Geometry*, p. 151. Spiral surfaces were first studied by Maurice Lévy in a paper published in Comptes Rendus, vol. 87 (1878), p. 788. He gave them as an example of surfaces such that E, F, G, the coefficients in the square of the linear element, are homogeneous functions of the parameters of any degree other than -2. Lévy named them "surfaces pseudo-moulures logarithmiques."

cular cone whose axis is the axis of the surface, cuts all elements of the cone at the same angle, and is projected on a plane perpendicular to the axis in a logarithmic spiral. Such a curve we shall call simply a spiral.* The equations of any spiral surface may be written

$$x = \rho e^{v} \cos(\omega + hv), \qquad y = \rho e^{v} \sin(\omega + hv), \qquad z = \zeta e^{v}, \tag{4}$$

where h is constant, ρ, ω, and ζ are functions of u alone, and the curves (u) are the spirals. Darboux has proved† that the orthogonal trajectories of the spirals of a spiral surface whose generating curve is given may be found by quadratures, and the square of the linear element written in the form

$$ds^2 = e^{2v} U^2 (du^2 + dv^2),$$

where U is a function of u alone. If U is given, h an arbitrary constant, and ζ, ρ, ω determined from

$$h^2(\zeta^2 + \zeta'^2) = h^2 U^2 - U'^2, \qquad (h^2+1)\rho^2 = U^2 - \zeta^2, \tag{5}$$

$$(h^2+1)\rho^2\omega' = -h\zeta\zeta' - \frac{1}{h} UU',$$

the surface (4) has the linear element in the form given, so that the solution of (5) determines a three-parameter family of spiral surfaces applicable to a given spiral surface.‡

Lie proved§ that all real minimal surfaces applicable to spiral surfaces are given by writing in (1)

$$F(u) = cu^{m-2+ni}, \qquad \Phi(v) = \bar{c}v^{m-2-ni}, \tag{6}$$

where c and $\bar{c}$ are conjugate constants and m and n are real constants. The number -2 is introduced in the exponents only to simplify future formulations. No essential restriction is imposed by taking the modulus of c as unity, so that we shall suppose $c = e^{\alpha i}$ where α is real. If in (6) $n = 0$ equations (1) and (6) give all minimal surfaces applicable to surfaces of revolution.‖ That these surfaces are applicable to surfaces of revolution if n is zero, and when n is not zero to spiral surfaces, appears from the linear element of (1) and (6) in terms of r, ϕ

$$ds^2 = e^{-2n\phi} r^{2m-4} (r^2+1)^2 (dr^2 + r^2 d\phi^2).$$

* These curves have been named "cylindro-conical helices." See *Encyclopädie der mathematischen Wissenschaften*, Bd. III, 3, Heft 2, 3, p. 252.

† L. c., pp. 148–150.

‡ Proved by Maurice Lévy, as stated by Darboux (l. c., p. 149). Of the three constants one is additive to ω and has no effect on the form of the surface, so that there are generally given by the integration of (5) a *two* parameter family of different surfaces.

§ Mathematische Annalen, vol. 15 (1879), p. 503. Darboux, l. c., p. 396.

‖ First proved by Schwarz, *Miscellen*, p. 296.

Evidently the Minding parallels (r) of (1) correspond to the spirals of the spiral surface, and, when n is zero, to the parallels of the surface of revolution. If $m = 0$, $n \neq 0$ equations (1) and (6) give all minimal surfaces which are spiral surfaces.*

A. Ribaucour proved† that the minimal surfaces associate to a minimal surface applicable to a surface of revolution are congruent to the given minimal surface except for $m = 0$, which value gives the minimal helicoids. Darboux states without proof that the minimal surfaces associate to a minimal surface applicable to a spiral surface are similar to the given surface. In the following section we prove these theorems, noting that exception must be made to Darboux's statement in the case of spiral minimal surfaces, and proving also a theorem, which we believe has not previously been stated, concerning the similarity to itself of the general surface given by (1) and (6). The remainder of this paper deals entirely with spiral minimal surfaces: we consider families of minimal surfaces associate to a surface of this kind, their Schwarz ellipses and the locus L of the vertices of these ellipses, the envelope of the family of associate surfaces, and the evolute surface; it is proved that on every spiral minimal surface there are an infinite number of plane spiral lines of curvature, the intersections of the surface with the plane part of L, an infinite number of spiral asymptotic lines, the intersections of the surface with the right conoids of L, and a single spiral geodesic; we show that on one member of a family of associate spiral minimal surfaces all the plane spiral lines of curvature except one are double curves; we discover certain symmetries of these surfaces, and finally obtain two characteristic properties of spiral minimal surfaces connected with the locus L and with the Minding parallels.

2. **Darboux's similarity theorem.** The first two of equations (1) may be combined and written

$$x + yi = -\int u^2 F(u)\,du + \int \Phi(v)\,dv.$$

For the minimal surfaces applicable to spiral surfaces we substitute from (6)

$$F(u) = e^{ai}\,u^{m-2+ni}, \qquad \Phi(v) = e^{-ai}\,v^{m-2-ni}, \qquad (n \neq 0).$$

When this surface is rotated about the Z axis through the angle β,

$$x' + y'i = e^{\beta i}(x + yi), \qquad z' = z,$$

where x', y', z' are the new coördinates of x, y, z. Sufficient conditions that this surface in the new position be similar to that surface for which

* First proved by Darboux, l. c., p. 359.

† "*Étude sur les élassoides ou surfaces à courbure moyenne nulle.*" Memoires Couronnés de l'Académie Royale de Belgique, vol. 44 (1882), chap. XX. Ribaucour disregards the cases $m = 0, \pm 1$; Darboux proves the theorem for all cases except $m = 0$ (l. c., pp. 395, 396).

$\alpha = 0$ in the original position with the origin as center, C as the ratio of similarity, and all constants of integration chosen as zero, are that u' and v' may be determined so that, with constant real values for β and C,

$$e^{\beta i}(-u'^2 F(u')\,du' + \Phi(v')\,dv') = C(-u^{m+ni}\,du + v^{m-2-ni}\,dv),$$
$$u' F(u')\,du' + v' \Phi(v')\,dv' = C(\ u^{m-1+ni}\,du + v^{m-1-ni}\,dv).$$

These equations are satisfied by constant real values of β and C by taking u' and v' as conjugate functions of u and v alone respectively if the same is true of the following four equations:

$$e^{(\alpha+\beta)i}\,u'^{m+ni}\,du' = Cu^{m+ni}\,du, \qquad e^{(\beta-\alpha)i}\,v'^{m-2-ni}\,dv' = Cv^{m-2-ni}\,dv,$$
$$e^{\alpha i}\,u'^{m-1+ni}\,du' = Cu^{m-1+ni}\,du, \qquad e^{-\alpha i}\,v'^{m-1-ni}\,dv' = Cv^{m-1-ni}\,dv.$$

From the last group of equations

$$u' = ue^{-\beta i}, \qquad v' = ve^{\beta i}.$$

Setting these values of u' and v' in the four equations, we have

$$C = e^{n\beta+(\alpha-m\beta)i}, \qquad C = e^{n\beta-(\alpha-m\beta)i},$$

so that we must have $\alpha - m\beta = 0$. If m is not zero the equations are satisfied by the constant real values,

$$\beta = \frac{\alpha}{m}, \qquad C = e^{n\alpha/m}. \tag{7}$$

If m is zero the proof fails. Equations (7) express the two theorems: (I) Minimal surfaces associate to a minimal surface applicable to a surface of revolution ($n = 0$) are congruent to that surface and may be made to coincide with it by a rotation about the Z axis, except in the case of the minimal helicoids ($m = 0$). That the associate minimal helicoids are not similar is well known.* (II) Minimal surfaces associate to a minimal surface applicable to a spiral surface ($n \neq 0$) are similar and are brought into similar position by a rotation about the Z axis, except in the case of the spiral minimal surfaces ($m = 0$). That the associates of a spiral minimal surface are not similar to that surface will appear in § 6 of this paper.

If we observe that the associate to any minimal surface given by $\alpha = 2k\pi$, where k is any integer, coincides with that surface, and that the associate $\alpha = (2k+1)\pi$ is symmetrical with respect to the origin to that surface, we have in (7) the proof of a theorem due to Ribaucour: A minimal surface applicable to a surface of revolution, not a minimal helicoid, is brought into coincidence with itself by rotation about the Z axis through the angle $2k\pi/m$; it appears also, though not stated by Ribaucour, that such a surface is brought

* See, for example, Scheffers, *Theorie der Flächen*, 2d ed., pp. 362, 363.

into coincidence with its symmetry with respect to the origin by rotation about the Z axis through the angle $(2k+1)\pi/m$. Finally it appears from (7) that a minimal surface applicable to a spiral surface ($m \neq 0$, $n \neq 0$) is similar to itself and to its symmetry with respect to the origin in an infinite number of positions if m is irrational, in a finite number if m is rational.

3. **Equations and linear element of $S_{n\alpha}$.** We denote by $S_{n\alpha}$ the spiral minimal surface given by substituting in (1)

$$F(u) = e^{\alpha i} u^{-2+ni}, \qquad \Phi(v) = e^{-\alpha i} v^{-2-ni}.$$

Performing the integrations indicated, taking all constants of integration as zero, then writing

$$u = re^{\phi i}, \qquad v = re^{-\phi i}, \qquad n = -\cot\beta,$$

we have the equations of $S_{n\alpha}$ in terms of the real parameters r, ϕ

$$\begin{aligned} x &= e^{-n\phi}\sin\beta\,[\,r\sin(\phi-\beta+\alpha+n\log r) \\ &\qquad\qquad + \frac{1}{r}\sin(\phi-\beta-\alpha-n\log r)\,], \\ y &= -\,e^{-n\phi}\sin\beta\,[\,r\cos(\phi-\beta+\alpha+n\log r) \\ &\qquad\qquad + \frac{1}{r}\cos(\phi-\beta-\alpha-n\log r)\,], \\ z &= -\,2e^{-n\phi}\tan\beta\sin(\alpha+n\log r). \end{aligned} \tag{8}$$

In these equations n is retained for the sake of abbreviation. If we let

$$n\log r = u, \qquad -n(\phi-\beta) = v, \qquad h = -\frac{1}{n}$$

and set

$$\begin{aligned} \rho\cos\omega &= 2e^{-n\beta}\sin\beta\sin(u+\alpha)\sinh\frac{u}{n}, \\ \rho\sin\omega &= -\,2e^{-n\beta}\sin\beta\cos(u+\alpha)\cosh\frac{u}{n}, \\ \zeta &= -\,2e^{-n\beta}\tan\beta\sin(u+\alpha), \end{aligned}$$

equations (8) take Darboux's form (4) thus proving that $S_{n\alpha}$ is actually a spiral surface and that the spirals are the Minding parallels (r).

The intersection of $S_{n\alpha}$ with the xy plane is given by

$$\alpha + n\log r = k\pi,$$

where k is any integer, and each such value of k gives a curve, as appears from the first two equations of (8), whose polar equation, coördinates R, Θ, is $R = e^{-n(\Theta+c)} = e^{(\Theta+c)\cot\beta}$, so that the surface cuts the plane in an infinite

number of congruent logarithmic spirals which cut all radii vectores at the angle β.

The linear element of (8) is given by

$$ds^2 = (r^2 + 1)^2 r^{-4} e^{-2n\phi} (dr^2 + r^2 d\phi^2).$$

Substituting the values u, v given after (8)

$$ds^2 = e^{2v} U^2 (du^2 + dv^2), \qquad U^2 = 4e^{-2n\beta} \tan^2 \beta \cosh^2 \frac{u}{n}.$$

We may determine the spiral minimal surfaces $S_{n\alpha}$ applicable to a given spiral minimal surface S_0 which are given by the integration of (5) as follows. Let the subscript apply to all quantities relating to S_0; from $U^2 = U_0^2$ follows $n^2 = n_0^2$; the first of equations (5) gives

$$\zeta^2 + \zeta'^2 = 4e^{-2n\beta} \tan^2 \beta = U_0^2 - U_0'^2/h^2.$$

In this equation we may replace U_0 by U, and are led to the condition $h^2 n^2 = h^2 n_0^2 = 1$. The solutions of (5) with $h = -1/n_0$ substituted in (4) give equations (8) where n is replaced by n_0, α is the constant introduced by the integration of the equation for ζ, and the constant additive to ω, which affects only the position of the surface, is taken as zero. The integration of (5) with $h = 1/n_0$ leads to the surfaces $S_{-n_0\alpha}$, surfaces not associate to S_0 but symmetrical with respect to the X axis to the associate surfaces, as will appear in § 9. Spiral surfaces, not minimal, applicable to S_0 would be given by the substitution in (4) of the solutions of (5) for values of h not equal to $\pm 1/n_0$, but it appears to be impossible to integrate the equation for ζ for such values.

4. **The locus L for $S_{n\alpha}$.** From equations (1) for S_{n0} we find

$$\sum V_1^2 / \sum U_1^2 = (uv)^{-2ni};$$

that part of L given by (3) where the first radical is taken as $(uv)^{-ni/2}$, which we call L_1, has the equations, obtained by substituting this value and writing as before $u = re^{\phi i}$, $v = re^{-\phi i}$,

$$(9) \qquad \begin{aligned} x &= e^{-n\phi} \frac{r^2+1}{r} \sin\beta \sin(\phi - \beta), \\ y &= -e^{-n\phi} \frac{r^2+1}{r} \sin\beta \cos(\phi - \beta), \\ z &= 0. \end{aligned}$$

Points common to L_1 and $S_{n\alpha}$ are given by

$$(uv)^{-ni/2} = e^{\alpha i} \qquad \text{or} \qquad \alpha + n \log r = 2k\pi.$$

The equations of L_2, symmetrical with respect to the origin to L_1, are

obtained by changing the signs of the second members of (9). Points common to L_2 and $S_{n\alpha}$ are given by $\alpha + n \log r = (2k+1)\pi$. The curves common to L_1, L_2, and $S_{n\alpha}$ are the logarithmic spirals in which the latter cuts the xy plane.

Choosing the first radical in (3) as $i(uv)^{-ni/2}$ we obtain for L_3, part of the locus L, the equations

$$(10)\qquad \begin{gathered} x = e^{-n\phi}\frac{r^2-1}{r}\sin\beta\cos(\phi-\beta), \qquad y = e^{-n\phi}\frac{r^2-1}{r}\sin\beta\sin(\phi-\beta), \\ z = -2e^{-n\phi}\tan\beta. \end{gathered}$$

Points common to L_3 and $S_{n\alpha}$ are given by

$$i(uv)^{-ni/2} = e^{\alpha i} \qquad \text{or} \qquad \alpha + n\log r = 2k\pi + \frac{\pi}{2}.$$

The equations of L_4, symmetrical to L_3, are found by reversing the signs of the second members of (10). Points common to L_4 and $S_{n\alpha}$ are given by $\alpha + n\log r = (2k+1)\pi + \pi/2$. Equations (9) and (10) may be obtained by substituting the appropriate values of r in (8) and regarding α as variable.

The surface L_3 is a spiral surface for (10) may be put in the form (4), and its spirals are the curves (r). From (10) it appears that every curve (ϕ) of L_3 is a straight line intersecting the Z axis and parallel to the xy plane, so that L_3 is a spiral right conoid.

The distances b of r, ϕ of L_1 from the origin and a of r, ϕ of L_3 from the origin are given respectively by

$$b^2 = e^{-2n\phi}\left(\frac{r^2+1}{r}\right)^2\sin^2\beta, \qquad a^2 = e^{-2n\phi}\left[\left(\frac{r^2-1}{r}\right)^2\sin^2\beta + 4\tan^2\beta\right],$$

and for all r, ϕ we have $a^2 > b^2$, so that a and b are respectively the semi-major and minor axes of Schwarz's ellipse r, ϕ; then the xy plane (L_1, L_2) is the locus of the extremities of the minor axes, and the two conoids (L_3, L_4) form the locus of the ends of the major axes. Since $a^2 - b^2$ is independent of r all ellipses corresponding to a constant ϕ have the same focal distance; the eccentricity is independent of ϕ so that all ellipses corresponding to points of a spiral (r) have the same eccentricity. The maximum eccentricity is $\sin\beta$ and is given by $r = 1$; all ellipses, $r = 1$, have their major axes on the Z axis and touch S_{n0} at the extremities of their minor axes. When r approaches zero or becomes infinite the eccentricity approaches zero, corresponding points of $S_{n\alpha}$ receding indefinitely from the origin.

5. **Surfaces related to $S_{n\alpha}$.** The *envelope* of a family of minimal surfaces associate to (1) consists of two surfaces symmetrical with respect to the origin

given by*

$$x = U_1 H + V_1 H^{-1}, \qquad y = U_2 H + V_2 H^{-1}, \qquad z = U_3 H + V_3 H^{-1},$$

where

$$H = \pm \sqrt{\frac{(u+v)V_1 + i(v-u)V_2 + (uv-1)V_3}{(u+v)U_1 + i(v-u)U_2 + (uv-1)U_3}}.$$

For E_1 and E_2, the two parts of the envelope of a family of associate spiral minimal surfaces, corresponding to the upper and lower signs respectively in H,

$$H = \pm e^{(\xi - n \log r)i}, \qquad \tan \xi = \frac{r^2 - 1}{r^2 + 1} \tan \beta .$$

The equations of E_1 are also given by writing $\xi - n \log r$ in place of α in (8). That E_1 is a spiral surface, whose spirals are the curves (r), may be shown by reducing its equations to the form (4); the expressions for $\rho \cos \omega$, $\rho \sin \omega$, and ζ for E_1 are obtained by replacing α by $\xi - u$ in the equations following (8). Points common to $S_{n\alpha}$ and E_1, $S_{n\alpha}$ and E_2 are given respectively by

$$\xi = \alpha + n \log r + 2k\pi, \qquad \xi = \alpha + n \log r + (2k+1)\pi .$$

Each of these equations has just one positive root r for every integral value of k, so that every surface $S_{n\alpha}$ is tangent to each surface of the envelope along an infinite number of spirals. The solution of the first equation for $\alpha = k = 0$ is $r = 1$, so that S_{n0} is tangent to E_1 along this curve, which will appear as a curve of particular interest.

The coördinates of the points of the two nappes of the *evolute surface* of $S_{n\alpha}$ corresponding to r, ϕ of the latter are

$$x \pm RX = x \pm e^{-n\phi} \frac{r^2 + 1}{r} \cos \phi ,$$

$$y \pm RY = y \pm e^{-n\phi} \frac{r^2 + 1}{r} \sin \phi ,$$

$$z \pm RZ = z \pm e^{-n\phi} \frac{r^4 - 1}{2r^2} ,$$

where x, y, z are the coördinates of r, ϕ on $S_{n\alpha}$ given by (8), X, Y, Z the direction cosines of the normal, and R the absolute value of either principal radius of curvature. The only interest in this evolute surface consists in the fact that it is a spiral surface, whose spirals are the curves (r), as may be seen by reducing its equations to the form (4).

In connection with the spiral surfaces related to $S_{n\alpha}$ the following considerations are of interest: the right circular cone whose axis is the Z axis

* My paper, American Journal of Mathematics, *l. c.*

and whose elements make the angle γ with this axis is

$$x^2 + y^2 = z^2 \tan^2 \gamma .$$

The spiral which cuts the elements of this cone under the angle λ is

$$x = e^{v \sin \gamma \cot \lambda} \sin \gamma \cos (c + v), \qquad y = e^{v \sin \gamma \cot \lambda} \sin \gamma \sin (c + v),$$
$$z = e^{v \sin \gamma \cot \lambda} \cos \gamma .$$

Equations (4) of any spiral surface may be written, setting $v = -n\phi$ and $h = -1/n$,

$$(4') \quad x = e^{-n\phi} \rho \cos (\omega + \phi), \qquad y = e^{-n\phi} \rho \sin (\omega + \phi), \qquad z = e^{-n\phi} \zeta .$$

The curve (r) of (4′) lies on the cone $x^2 + y^2 = \rho^2 z^2/\zeta^2$, so that $\tan \gamma = \rho/\zeta$, which gives, for $r = 1$, $\tan \gamma = \cos \beta \cot \alpha$. We find

$$\tan \lambda = \sin \gamma \tan \beta ,$$

so that λ is determined by $n = -\cot \beta$ and γ. Since for fixed n (4′) includes the surfaces $S_{n\alpha}$, L_3, E_1, and the evolute surface of $S_{n\alpha}$, it follows that every right circular cone whose axis is the Z axis cuts all these surfaces in congruent spirals.

6. **Special spirals on $S_{n\alpha}$.** The differential equation of the *lines of curvature* of (1) becomes for $S_{n\alpha}$

$$e^{\alpha i} u^{-2+ni} du^2 - e^{-\alpha i} v^{-2-ni} dv^2 = 0 .$$

This equation integrated gives

$$e^{\alpha i} (uv)^{ni/2} \pm 1 = cv^{ni/2} .$$

The only spirals (r) among the lines of curvature are given by $c = 0$. For them we have, belonging respectively to the two families of lines of curvature,

$$\alpha + n \log r = 2k\pi , \qquad \alpha + n \log r = (2k + 1)\pi .$$

These are the curves of intersection of $S_{n\alpha}$ with the xy plane, and are logarithmic spirals; they are moreover the curves common to $S_{n\alpha}$ and L_1, $S_{n\alpha}$ and L_2 respectively. For $\alpha = k = 0$ the first equation gives $r = 1$, which is therefore a line of curvature of S_{n0}.

The differential equation of the *asymptotic lines* of $S_{n\alpha}$ is

$$e^{\alpha i} u^{-2+ni} du^2 + e^{-\alpha i} v^{-2-ni} dv^2 = 0 ,$$

giving the integral

$$e^{\alpha i} (uv)^{ni/2} \pm i = cv^{ni/2} .$$

The only spirals among the asymptotic lines are given by $c = 0$; for these,

belonging respectively to the two families of asymptotic lines,

$$\alpha + n \log r = 2k\pi + \frac{\pi}{2}, \qquad \alpha + n \log r = (2k+1)\pi + \frac{\pi}{2}.$$

These are the curves common to $S_{n\alpha}$ and L_3, $S_{n\alpha}$ and L_4 respectively. The first of these equations gives $r = 1$ for $\alpha - \pi/2 = k = 0$. This line on $S_{n\pi/2}$ is the Z axis. Since on every surface the Minding parallels with reference to the Z axis and the curves of steepest ascent form a conjugate system,* the asymptotic spirals are curves of steepest ascent and are the only such spirals.

The properties of the Minding parallels (r) of $S_{n\alpha}$ are analogous to properties of the Minding meridians (ϕ) of minimal surfaces applicable to surfaces of revolution.† We note further that $S_{n\alpha}$ has no line of curvature or asymptotic line (ϕ) while minimal surfaces applicable to surfaces of revolution, with the exception of the catenoid and its adjoint surface, the right helicoid, have no line of curvature or asymptotic line (r).

From the linear element of $S_{n\alpha}$, given in § 3, it appears‡ that the only *spiral geodesic* of this surface is $r = 1$. This spiral being plane for $\alpha = 0$, S_{n0} has symmetry with respect to its plane, the xy plane; since, for $\alpha = \pi/2$, this spiral is the Z axis $S_{n\pi/2}$ has symmetry with respect to this axis. On any surface similar to $S_{n\alpha}$ there will correspond to the spiral geodesic, $r = 1$, a spiral geodesic on a cone whose semi-vertical angle γ is given by

$$\tan \gamma = \cos \beta \cot \alpha.$$

On the associate surface $S_{n\alpha'}$ the only spiral geodesic lies on a cone of semi-vertical angle γ' given by $\tan \gamma' = \cos \beta \cot \alpha'$. Then $S_{n\alpha}$ and $S_{n\alpha'}$ are not similar if $\alpha' - \alpha \neq k\pi$. There is no geodesic (ϕ) of $S_{n\alpha}$.

7. **Double spirals of S_{n0}.** Writing $\alpha = 0$ in (8) the equations of S_{n0} are

$$x = e^{-n\phi} \sin \beta \left[r \sin(\phi - \beta + n \log r) + \frac{1}{r} \sin(\phi - \beta - n \log r) \right],$$

$$y = - e^{-n\phi} \sin \beta \left[r \cos(\phi - \beta + n \log r) + \frac{1}{r} \cos(\phi - \beta - n \log r) \right],$$

$$z = - 2e^{-n\phi} \tan \beta \sin(n \log r).$$

If in these equations r is replaced by $1/r$, x and y are unchanged and the sign of z is reversed. If we consider those values of r satisfying the equation, $n \log r = k\pi$, we have the logarithmic spiral lines of curvature in which S_{n0}

* My paper, Annals of Mathematics, ser. 2, vol. 19, p. 4.

† My paper, Annals of Mathematics, l. c.

‡ Eisenhart, *Differential Geometry*, pp. 134, 266, 267.

intersects the xy plane; excepting the value $r = 1$, it appears that each of these spirals is given by two different reciprocal values of r. The direction cosines of the normal to the surface, given in § 1, are in terms of r, ϕ

$$X = \frac{2r\cos\phi}{r^2+1}, \qquad Y = \frac{2r\sin\phi}{r^2+1}, \qquad Z = \frac{r^2-1}{r^2+1}.$$

Changing r to $1/r$, X and Y are unaltered and the sign of Z is reversed. It follows that along each spiral line of curvature of S_{n0}, except $r = 1$, two branches of the surface intersect; moreover they intersect at a constant angle along each curve, the angle depending on r alone and approaching the limit π as r increases indefinitely from unity.

8. **Further properties of spirals on $S_{n\alpha}$.** In the first paper* published dealing with minimal surfaces applicable to surfaces of revolution E. Bour showed that every Minding meridian (ϕ) of the surface cuts at the same angle all members of either family of lines of curvature and all level curves (z). The Minding parallels (r) have the same property on $S_{n\alpha}$, as may be shown directly; we find, for example, that the angle of (r) and (z) is $\alpha + n\log r$, and is therefore constant with r.

The equation of the tangent plane to $S_{n\alpha}$ at r, ϕ is

$$x\cos\phi + y\sin\phi + z\frac{r^2-1}{2r} = P(r)e^{-n\phi},$$

$$P(r) = -\frac{r^2+1}{r}\sin^2\beta\sec\xi\cos(\alpha + n\log r - \xi),$$

where ξ is the angle introduced in § 5. The helicoidal developable tangent to $S_{n\alpha}$ along (r) therefore intersects the xy plane in the envelope of the lines, $x\cos\phi + y\sin\phi = P(r)e^{-n\phi}$, which is a logarithmic spiral congruent to the plane lines of curvature (r). It may also be shown from the equation of the tangent plane that the planes tangent to the surface along (ϕ) envelop a cylinder whose elements are parallel to the xy plane, and that the cylinders corresponding to different values of ϕ are similar and are brought into similar position by rotation about the Z-axis through the angle ϕ. Bour stated in the paper cited that in the case of a minimal surface applicable to a surface of revolution the developables tangent to the surface along a Minding meridian (ϕ) are cylinders whose elements are parallel to the xy plane; we have proved that this is a property of every non-developable surface.† Ribaucour proved in the *mémoire* cited that in the case of minimal surfaces applicable to surfaces

* "*Théorie de la déformation des surfaces*," Journal de l'école polytechnique, cahier 39 (1862).

† Annals of Mathematics, l. c.

of revolution these cylinders are similar. Stübler* determined all minimal surfaces which are envelopes of similar cylinders which may be brought into similar position by rotation about the Z-axis, finding that such surfaces are given by three different forms of $F(u)$ in (1), each depending on several parameters, these surfaces including those applicable to surfaces of revolution and spiral minimal surfaces.

9. **Symmetries of $S_{n\alpha}$.** From the equations (8) of $S_{n\alpha}$ it appears that if α and r are replaced by $-\alpha$ and $1/r$ respectively the values of x and y are unchanged and the sign of z is reversed; it follows that $S_{n\alpha}$ and $S_{n-\alpha}$ have symmetry with respect to the xy plane, symmetrical points being given by r, ϕ and $1/r, \phi$. If in (8) n is changed to $-n$, and consequently β to $-\beta$, if then r, ϕ are replaced by $1/r, -\phi$ respectively, x is unchanged, and the signs of both y and z are reversed; then $S_{n\alpha}$ and $S_{-n\alpha}$ are symmetrical with respect to the X axis. It follows that $S_{n\alpha}$ and $S_{-n-\alpha}$ are symmetrical with respect to the xz plane. If the isothermal parameters $\log r, \phi$ are used as coördinates symmetrical points in all three cases are given by changing the signs of one or of both coördinates.

10. **Two properties characteristic of $S_{n\alpha}$.** We prove a characteristic property of spiral minimal surfaces together with a similar characteristic property of minimal surfaces applicable to surfaces of revolution given by the following theorem, a converse of theorems given in § 8. If every Minding parallel (r) of a real minimal surface is an isogonal trajectory of either family of lines of curvature or of the level curves of the surface the minimal surface is a spiral surface or a helicoid; if every Minding meridian (ϕ) of a real minimal surface is an isogonal trajectory of either of the families named the minimal surface is applicable to a surface of revolution.

To prove the first part of this theorem we observe that the condition that all curves (r) be isogonal trajectories of either of the families named may be expressed by the equation

$$\frac{u^2 F(u)}{v^2 \Phi(v)} = f(uv),$$

where F and Φ are the conjugate functions in (1) and f is unknown. If this equation is differentiated first with respect to u and then with respect to v, and the two values of f' equated, we have

$$\frac{2F(u) + uF'(u)}{F(u)} = -\frac{2\Phi(v) + v\Phi'(v)}{\Phi(v)} = ni,$$

where n is a real constant. These give

$$F(u) = cu^{-2+ni}, \qquad \Phi(v) = \bar{c}v^{-2-ni},$$

*Mathematische Annalen, vol. 75 (1914).

so that the minimal surface is a spiral surface, or, if $n = 0$, a helicoid. The condition that each curve (ϕ) of (1) be an isogonal trajectory of any one of the families named is

$$\frac{u^2 F(u)}{v^2 \Phi(v)} = f\left(\frac{u}{v}\right),$$

which leads in a similar way to

$$F(u) = cu^{m-2}, \qquad \Phi(v) = \bar{c}v^{m-2},$$

where m is a real constant. The only real minimal surfaces such that both families of curves (r) and (ϕ) have the isogonal property are the helicoids. It is easily shown that it is only on the minimal helicoids that either of these families, (r) and (ϕ), forms an isogonal *system* with any of the three other families named, for only in this case is the angle of intersection constant.

It appeared in § 4 that for spiral minimal surfaces part of the locus L is a plane containing the center of the Schwarz ellipses;* we have elsewhere proved that the same thing is true for minimal surfaces applicable to surfaces of revolution if m is different from zero or plus or minus one. We now prove that this property is characteristic of these two classes of real minimal surfaces.

Since the equations of any minimal surface in any position are given by (1) we may without restriction suppose the equations of a minimal surface for which part of L is a plane containing the center of the Schwarz ellipses to have the form (1) and this plane part of L to be the xy plane, $z = 0$. From (3) it follows that, for all u, v,

$$z = U_3\sqrt[4]{\frac{\sum V_1'^2}{\sum U_1'^2}} + V_3\sqrt[4]{\frac{\sum U_1'^2}{\sum V_1'^2}} = 0,$$

for some determination of the first radical. This gives one of the identities

$$\frac{U_3}{\sqrt{\sum U_1'^2}} = \frac{V_3}{\sqrt{\sum V_1'^2}} = c \qquad \text{or} \qquad \frac{U_3}{\sqrt{\sum U_1'^2}} = -\frac{V_3}{\sqrt{\sum V_1'^2}} = c,$$

where $\sqrt{\sum U_1'^2}$ and $\sqrt{\sum V_1'^2}$ are conjugate and c is a constant real in the first case and pure imaginary in the second. We consider the equation

$$U_3/\sqrt{\sum U_1'^2} = c,$$

and write

$$\sqrt{\sum U_1'^2} = \rho, \qquad U_1 = \lambda\rho, \qquad U_2 = \mu\rho, \qquad U_3 = c\rho.$$

From the last equations

$$\lambda^2 + \mu^2 + c^2 = 1, \qquad \lambda\lambda' + \mu\mu' = 0,$$

$$U_1' = \lambda\rho' + \lambda'\rho, \qquad U_2' = \mu\rho' + \mu'\rho, \qquad U_3' = c\rho'.$$

*American Journal of Mathematics, l. c.

From the fact that $u + v$, $i(v - u)$, $uv - 1$ are proportional to the direction cosines of the normal to (1) it follows that for all u, v

$$(u + v)U_1' + i(v - u)U_2' + (uv - 1)U_3' = 0,$$

which may be replaced by the two equations

$$u(U_1' - iU_2') - U_3' = 0, \qquad U_1' + iU_2' + uU_3' = 0.$$

The last two equations give $\sum U_1'^2 = 0$, from which, substituting the values given for U_1', U_2', U_3'

$$\rho'^2 + \rho^2(\lambda'^2 + \mu'^2) = 0, \qquad \rho = Ce^{\pm i\int\sqrt{\lambda'^2+\mu'^2}du}.$$

Substituting the values of U_1', U_2', U_3' in the two linear equations connecting these quantities, then replacing in these ρ' by $\pm \rho i\sqrt{\lambda'^2 + \mu'^2}$, and cancelling ρ, which cannot be identically zero unless (1) is a plane,* we find

$$\frac{\lambda' + i\mu'}{\mp\sqrt{\lambda'^2 + \mu'^2}} = i(\lambda + \mu i + cu), \qquad \frac{\lambda' - i\mu'}{\mp\sqrt{\lambda'^2 + \mu'^2}} = i\left(\lambda - \mu i - \frac{c}{u}\right).$$

Multiplying the last two equations,

$$(\lambda + i\mu + cu)\left(\lambda - i\mu - \frac{c}{u}\right) + 1 = 0.$$

From this and $\lambda^2 + \mu^2 + c^2 = 1$ we find

$$\lambda + i\mu = \frac{1 - c^2 \pm \sqrt{1 - c^2}\,u}{c}, \qquad \lambda - i\mu = -\frac{1 - c^2 \mp \sqrt{1 - c^2}}{cu}.$$

Differentiating and multiplying the results, $\lambda'^2 + \mu'^2 = (c^2 - 1)/u^2$, and

$$\rho = Ku^{\pm\sqrt{1-c^2}}, \qquad U_3 = c\rho = cKu^{\pm\sqrt{1-c^2}}.$$

If $c = 0$ the surface (1) is the plane, $z = 0$; if $c = \pm 1$ the surface is the plane $z =$ constant; for all other values of c the function $F(u)$ of (1) is given by

$$F(u) = \frac{1}{u}U_3' = cKu^{-2\pm\sqrt{1-c^2}}.$$

If c is pure imaginary we have $F(u) = Au^{m-2}$, where m is a real constant numerically greater than one; if c is real and numerically less than one, $F(u)$ has the same form and m is real and numerically less than one. Such values of c give all minimal surfaces applicable to surfaces of revolution, except that

* My paper, American Journal of Mathematics, l. c.

the values $m = 0, \pm 1$ are excluded, and distinguish two classes of these surfaces which have widely different properties.* Finally, if c is real and numerically greater than one, $F(u) = Au^{-2+ni}$, where n is a real constant, and the surface (1) is a spiral surface.

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* My paper, Annals of Mathematics, l. c.

ON THE GROUP OF ISOMORPHISMS OF A CERTAIN EXTENSION OF AN ABELIAN GROUP*

BY

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Introduction

In 1908 Professor G. A. Miller showed that "if an abelian group H which involves operators whose orders exceed 2 is extended by means of an operator of order 2 which transforms each operator of H into its inverse, then the group of isomorphisms of this extended group is the holomorph of H."† The present paper discusses an elaboration of the idea embodied in Professor Miller's thoerem, the successive developments taking an abelian group H each of more general type so that *in toto* it is proved that if G is formed by extending H which has operators of order > 2 by a *certain* operator from its group of isomorphisms which transforms every one of its operators into the same power of itself and which is commutative with no operator of odd order in it, then the group of isomorphisms of G is the holomorph of H, and is a complete group if H is of odd order. If H contains no operators of even order, the "certain operator" is any operator from the group of isomorphisms of H effecting the stated automorphism; if H contains no operator of odd order, the "certain operator" must transform every operator of H into its inverse; if H contains operators of both even and odd orders (*a*) the order of the automorphism of the operators of odd order effected by the extending operator is to be divisible by 2^n where H contains an operator of order 2^n but none of order 2^{n+1}, or else (*b*) the extending operator transforms into its inverse every one of H's operators whose order is a power of 2. Obviously it is not necessary that the extending operator be "from its group of isomorphisms," but that it have certain properties possessed by this operator; thus, besides effecting the automorphism required, its first power commutative with all the operators of H must be the identity (from which follows that its first power appearing in H is the identity).

The general method used in establishing each of the successive theorems is the same; two important steps in each proof are showing that H is character-

* Presented to the Society, September 5, 1917.

† Miller, The Philosophical Magazine, vol. 231 (1908), p. 224.

istic in G and that the group of isomorphisms of G contains an invariant subgroup simply isomorphic with H. Throughout the paper use is frequently made of properties of rational integers, these interrelationships being but simple illustrations of the comradeship existing between group theory and number theory. The group of isomorphisms is represented by I and a rational prime (odd unless otherwise specified) by p, and the statement "t transforms every operator of a group into the same power of itself, not the first power" means every operator except the identity.

Theory

1. Theorem 1. *If a group G is formed by extending a cyclic group H of order p^m (p an odd prime) by an operator from its group of isomorphisms which transforms every one of its operators into the nth power of itself where $n \not\equiv 1$, mod p, then the I of G is the holomorph of H and is a complete group.*

Let s be a generator of H and let r be the order of the extending operator t (note that r divides $\phi(p^m)$, $= p^{m-1}(p-1)$*); then r is the exponent to which n appertains, mod p^m [Note 1]; and t is from the central of the I of H.†

If s_j is any operator of H, $t^{-1} s_j t = s_j^n$, and ts_j effects the same automorphism of H as does t, while $t^a s_j$, $1 < a < r$ effects a different one. Moreover, ts_j is of order r, for

$$
\begin{aligned}
(ts_j)^r &= t^{r+1} \cdot t^{-r} s_j t^r \cdot \quad \cdots \quad \cdot t^{-2} s_j t^2 \cdot t^{-1} s_j t \cdot t^{-1}\ddagger \\
&= t(s_j^n)^{\frac{n^r-1}{n-1}} t^{-1} = [ts_j^n t^{-1}]^{\frac{n^r-1}{n-1}} \\
&= s_j^{\frac{n^r-1}{n-1}} \text{ (since } t^{-1} s_j t = s_j^n \text{ gives } s_j = ts_j^n t^{-1}) \\
&= 1 \text{ (since } n^r - 1 \equiv 0, \text{ mod } p^m \text{ and } n - 1 \not\equiv 0, \text{ mod } p\P).
\end{aligned}
$$

Furthermore, ts_j cannot be of order $< r$, for *if t is an operator from the I of a group H, no other operator effecting the same automorphism of H can be of lower order,* because the order of an operator in the I of a group must be exactly the order of the automorphism it effects. From these properties of ts_j it is evident that, as soon as H has been shown characteristic in G, in all the automorphisms of G t corresponds only to itself or to itself multiplied by some operator of H.

It will next be shown that the orders of the operators in the tail of G either divide r or divide p^m.§ Let $t^a s_j$ be such an operator, $1 \leqq a < r$ which is $\leqq \phi(p^m)$. The cases will be considered separately.

* Mathews, *Theory of Numbers* (1892), p. 18.

† Miller, these Transactions, vol. 1 (1900), p. 397.

‡ Miller, Blichfeldt, and Dickson, *Finite Groups* (1916), § 24.

¶ Cf. Gauss, *Disquisitiones Arithmeticæ* (1801), § 79.

§ Cf. Miller, these Transactions, vol. 4 (1903), p. 156.

Case I. If $n^a \not\equiv 1$, mod p, the order of $t^a s_j$ divides r.

$$(t^a s_j)^r = t^{a(r+1)} \cdot t^{-ra} s_j t^{ra} \cdot \quad \cdots \quad \cdot t^{-2a} s_j t^{2a} \cdot t^{-a} s_j t^a \cdot t^{-a}$$

$$= t^a (s_j^{n^a})^{\frac{n^{ra}-1}{n^a-1}} t^{-a} = [t^a s_j^{n^a} t^{-a}]^{\frac{n^{ra}-1}{n^a-1}}$$

$= s_j^{\frac{n^{ra}-1}{n^a-1}}$ (since $t^{-1} s_j t = s_j^n$ gives $t^{-a} s_j t^a = s_j^{n^a}$, whence $s_j = t^a s_j^{n^a} t^{-a}$) $= 1$, because $n^r = 1$, mod p^m, so that $n^{ra} - 1$ is divisible by p^m while in this first case $n^a - 1$ is not divisible by p.

Case II. If $n^a \equiv 1$, mod p, the order of $t^a s_j$ divides p^m.

$$(t^a s_j)^{p^m} = t^a \cdot t^{-ap^m} s_j t^{ap^m} \cdot \quad \cdots \quad \cdot t^{-2a} s_j t^{2a} \cdot t^{-a} s_j t^a \cdot t^{-a}$$

$= t^a (s_j^{n^a})^{\frac{n^{ap^m}-1}{n^a-1}} t^{-a} = s_j^{\frac{n^{ap^m}-1}{n^a-1}} = 1$. For suppose p^b $(b > 0)$ is the highest power of p dividing $n^a - 1$; then $n^a \equiv 1$, mod p^b, and $(n^a)^{p^m} \equiv 1$, mod p^{m+b} [Note 2]. Hence,

$$\frac{n^{ap^m} - 1}{n^a - 1} \equiv 0, \qquad \text{mod } p^m.$$

NOTE 1. *If p is an odd prime and $n \not\equiv 1$, mod p, and if n appertains to the exponent q, mod p^m, then it appertains, mod p^{m+1}, to q or else to pq.*

Obviously the exponent to which n appertains, mod p^{m+1}, is not less than q. Let $n^q = l + kp^m$. Two cases will be made.

If $k \equiv 0$, mod p, evidently n appertains to the exponent q, mod p^{m+1}.

If $k \not\equiv 0$, mod p, let n appertain to exponent v, mod p^{m+1}, or $n^v = l + lp^{m+1}$.

Subtracting the preceding equation gives $n^q(n^{v-q} - 1) = (lp - k)p^m$. Since neither n^q nor $(lp - k)$ is divisible by p, $n^{v-q} - 1$ is divisible by p^m but not by p^{m+1}, so that $v - q \equiv 0$, mod q, or v is a multiple of q. On raising n^q to the ath power there results $n^{aq} = (l + kp^m)^a = l + akp^m + Np^{m+1}$ (where N is a positive integer). From the last member, evidently the first value a can take so that n^{aq} shall be $\equiv 1$, mod p^{m+1} is the value p. Hence the proposition.

COROLLARY. *If p is an odd prime and $n \not\equiv 1$, mod p, and if n appertains to the exponent q, mod p^m, then it appertains, mod p^{m-1}, to q or else to q/p.*

For the even prime it is similarly demonstrable that *if $n \equiv 1$, mod 2, and n appertains to the exponent 2^u, mod 2^m, then it appertains, mod 2^{m+1}, to 2^u or else to 2^{u+1}.*

COROLLARY. *If $n \equiv 1$, mod 2, and if n appertains to the exponent 2^u, mod 2^m, then it appertains, mod 2^{m-1}, to 2^u or else to 2^{u-1}.*

NOTE 2. *If $d \equiv 1$, mod p^e, where p is an odd prime $(e > 0)$, then $d^{p^m} \equiv 1$, mod p^{e+m}.* This is readily established by mathematical induction. It is evidently true for $m = 0$. Assume true for mth power of p; i. e., $d^{p^m} = 1 + kp^{e+m}$. Raising to the pth power gives $d^{p^{m+1}} = 1 + (k + M)p^{e+m+1}$, ($M$ is an integer divisible by p) or $d^{p^{m+1}} \equiv 1$, mod p^{e+m+1}. Hence the truth of the proposition.* Furthermore, if p^e is the highest power of p dividing $d - 1$, then p^{e+m} is the highest power of p dividing $d^{p^m} - 1$; because, if mathematical induction is again employed, it will be observed that k would be prime to p while M is a multiple of p.

The proof is similar for the even prime that *if $d \equiv 1$, mod 2^e, $(e > 0)$, then $d^{2^m} \equiv 1$, mod 2^{e+m}.* Similarly, if 2 is the highest power of 2 dividing $d - 1$, then $d^{2^m} - 1$ $(m > 0)$, is divisible by at least 2^{m+2}; and if 2^e, $e > 1$, is the highest power of 2 dividing $d - 1$, then 2^{e+m} is the highest power of 2 dividing $d^{2^m} - 1$, $(m \geqq 0)$.

* Cf. Dirichlet, *Vorlesungen über Zahlentheorie* (1879), p. 333.

From the preceding it is further evident that if $r \geqq p - 1$, a can always take a value such that $n^a \equiv 1$, mod p. Then when s_j is of order p^m, or is s, $t^a s$ will likewise be of order p^m; because (as has just been shown) its order divides p^m, and its p^{m-1}th power is

$$(t^a s)^{p^{m-1}} = t^{ap^{m-1}} \cdot t^a (s^{n^a})^{\frac{n^{ap^{m-1}}-1}{n^a-1}} t^{-a} = t^{ap^{m-1}} s^{\frac{n^{ap^{m-1}}-1}{n^a-1}},$$

in which $(n^{ap^{m-1}} - 1)/(n^a - 1)$ is divisible by p^{m-1} but not by p^m. [Note 2.]

Next, in order to show that H is characteristic in G it will be proved that the cyclic subgroup of order p^m generated by an operator of order p^m, such as $t^a s$, from the tail of G is not an invariant subgroup of G. (Note that in $t^a s$, s is of order p^m and $n^a \equiv 1$, mod p.) Transforming by t should, if this subgroup is invariant in G, give an operator commutative with $t^a s$; i. e., $t^{-1}(t^a s)t = t^a s^n$, which if commutative with $t^a s$, requires that $t^a st^a s^n = t^a s^n t^a s$, or $st^a s^n = s^n t^a s$, or $t^a s^{n-1} = s^{n-1} t^a$. But this cannot be satisfied since the rth power of t is the first power of t commutative with s and since by hypothesis $n - 1$ is not divisible by p. Accordingly, H is the only invariant cyclic subgroup of order p^m in G and is therefore characteristic.

Since any automorphism of G is determined by an automorphism of the characteristic subgroup H and some one of the p^m operators of order r (that is, t, ts, ts^2, etc.), the I of G can be represented as a transitive substitution group on p^m letters. Now the I of G contains an invariant cyclic subgroup E simply isomorphic with H. For, suppose v is an operator from the I of G leaving the operators of H invariant but transforming t into ts, where s is of order p^m. Then $v^{-1} tv = ts$ gives $v^{-2} tv^2 = ts^2$, etc., from which it is seen that v is of the same order as s. Hence E is a cyclic subgroup of order p^m.

The characteristic subgroup H can be automorphic exactly as any cyclic group of order p^m, because t transforms every operator of H into the same power of itself and because the first power of t appearing in H is the identity. Hence, the order of the I of G is equal to the order of H multiplied by the order of its own group of isomorphisms, which product is the order of the holomorph of H.* Furthermore, since the I of G contains an invariant cyclic subgroup of order p^m and can be written transitively on p^m letters, it is simply isomorphic with the holomorph of H. This last point can be proved directly by showing that operators effecting automorphisms of the characteristic subgroup H, transform the operators of the invariant cyclic subgroup E in exactly the same way. Since H is of odd order, the I of G has the interesting property of being complete, since "the holomorph of any abelian group of odd order is a complete group."†

2. If the order of the cyclic group H is $2p^m$, H contains a characteristic

* Miller, Blichfeldt, and Dickson, loc. cit., p. 46.

† Miller, Mathematische Annalen, vol. 66 (1908), p. 135.

operator of order 2, furthermore G (formed by extending H by an operator from its group of isomorphisms which transforms every one of its operators of order > 2 into the nth power of itself where $n \not\equiv 1$, mod p) would be the direct product of two characteristic subgroups having only the identity in common, one this group of order 2 and the other the non-abelian group formed by extending the cyclic subgroup (of H) of order p^m. The I of this G is thus the holomorph of H.*

3. Now a cyclic group H of order > 2 can be extended by an operator of order 2 which transforms its operators into their inverses, and the group of isomorphisms of the extended group is the holomorph of H. If h (which is the order of the group H) is a number > 6 and has primitive roots, evidently H can be extended by operators from its group of isomorphisms that will transform every operator of order > 2 into the same power of itself other than into its inverse. To have primitive roots h must be of the form $2, 4, p^m$, or $2p^m$ (p an odd prime).† As the case $h = 2$ is trivial and $h = 4$ comes under the theorem on inverses, h will be considered to be one of the other forms; that is, $h = p^m$ or $2p^m$. Then since a primitive root n of h ($m > 1$) is a primitive root of all powers of p,‡ the following may be based upon what has just been proved or can be proved independently:

Corollary. *If a group G is formed by extending a cyclic group H whose order $h = p^m$ or $2p^m$ by an operator from its group of isomorphisms which transforms every one of its operators of order > 2 into the nth power of itself where n is a primitive root of h, then the I of G is simply isomorphic with G, moreover G is complete if $h = p^m$.*

Here n appertains to the exponent $\phi(h)$, mod h; hence r, the order of the extending operator t, is $\phi(h)$, and the order of G is $h\phi(h)$ so that G itself is seen to be the holomorph of H. H is the only invariant cyclic subgroup of order h in G and is therefore characteristic. If $h = p^m$, G is the holomorph of an abelian group of odd order which is characteristic in it, and hence G is complete.§

4. In case the cyclic group H is of order 2^m ($m > 1$) and $n \equiv 1$, mod 2, r, the order of the extending operator t from the group of isomorphisms of H, divides $\phi(2^m) = 2^{m-1}$ (and if $m > 2$, r is not greater than 2^{m-2}).‖ The attempt to establish a general theorem analogous to the preceding by a similar

* Miller, these Transactions, vol. 1 (1900), p. 396.

† Mathews, loc. cit., §§ 19–29.

‡ Lebesgue, Journal de Mathematiques, vol. 19 (1854), p. 334; cf. also Dirichlet, loc. cit., p. 334.

§ Burnside, *Theory of Groups* (1897), § 169; cf. Miller, Mathematische Annalen, vol. 66 (1908), p. 135.

‖ Weber, *Lehrbuch der Algebra*, Bd. II (1899), § 17; cf. Burnside, loc. cit., also Miller, Bulletin of the American Mathematical Society, vol. 7 (1901), p. 351.

method fails because ts (where s is a generator of H) is not of the same order as t, excepting in the case $t^{-1}st = s^{-1}$, when G is dihedral.* For, suppose $t^{-1}st = s^n$ where the order of t is 2^u, so that n appertains to 2^u, mod 2^m (whence $u < m$). It will be shown that ts is of order 2^u when and only when $n = 2^m - 1$, which means that t transforms the operators of H into their respective inverses so that its order is 2 and $u = 1$.

$$(ts)^{2u} = t \cdot t^{-2u} st^{2u} \cdot \quad \cdots \quad \cdot t^{-2} st^2 \cdot t^{-1} st \cdot t^{-1}$$

$$= t(s^n)^{\frac{n^{2^u}-1}{n-1}} t^{-1} = s^{\frac{n^{2^u}-1}{n-1}},$$ which if equal to the identity, requires that $(n^{2^u} - 1)/(n - 1)$ be congruent to zero, mod 2^m. In Note 3 it is shown by number theory that this is true when and only when $n = 2^m - 1$. It may be observed, however, that by use of Note 2 it can be proved easily that the order of $t^a s^c$ divides 2^m.

5. It is now an easy step from Theorem 1 to the following (in which p still represents an odd prime):

THEOREM 2. *If a group G is formed by extending an abelian group H of order p^m, type $(m_1, m_2, \cdots, m_k)$, $m_1 \geqq m_2 \geqq \cdots \geqq m_k$, $m = m_1 + m_2 + \cdots + m_k$, by an operator from its group of isomorphisms which transforms every one of its operators into the nth power of itself where $n \not\equiv 1$, mod p, then the I of G is the holomorph of H and is a complete group.*

NOTE 3. *If n appertains, mod 2^m, to 2^u $(u > 0)$, then a necessary and sufficient condition that*

$$\frac{n^{2^u} - 1}{n - 1} \equiv 0,$$

mod 2^m, is that $n = 2^m - 1$. (It is supposed that $m > 1$, that n is odd and $0 < n < 2^m$).

If $n = 2^m - 1$, then $u = 1$, and substituting shows that this value is sufficient to satisfy the congruence.

It will now be shown that it is necessary that $u \not> 1$ and from this that $n = 2^m - 1$. Let the highest power of 2 dividing $n - 1$ be 2^e. Now e must $= 1$, for if $e > 1$, from Note 2 the highest power of 2 dividing $(n^{2^u} - 1)/(n - 1)$ would be 2^u, which is less than 2^m since 2^u divides $\phi(2^m)$. Accordingly, $e = 1$ and $n = 1 + 2k$ where k is odd. The requirement that $(n^{2^u} - 1)/(n - 1)$ be divisible by 2^m then requires $n^{2^u} - 1$ to be divisible by 2^{m+1}. Squaring $n = 1 + 2k$ gives $n^2 = 1 + k(k + 1)2^2$; hence $n^2 - 1$ is divisible by 2^{b+2} (and by no higher power of 2) where 2^b $(b > 0)$ is the highest power of 2 dividing $k + 1$; and hence by Note 2, $n^{2^u} - 1$ is divisible by 2^{b+u+1} (and by no higher power of 2), so that the requirement that $n^{2^u} - 1$ be divisible by 2^{m+1} would necessitate $b + u + 1 \geqq m + 1$, or $b + u \geqq m$. Again by Note 2, if $u > 1$, $n^{2^{u-1}} - 1$ is divisible by 2^{b+u}, and hence by 2^m (because $b + u \geqq m$). But this is impossible since n appertains to exponent 2^u, mod 2^m. Therefore $u \not> 1$, or $u = 1$. Then $b \geqq m - 1$, so that

$$n = 1 + 2k = 2(k + 1) - 1 = 2(c \cdot 2^{m-1}) - 1 = c \cdot 2^m - 1.$$

Since $n < 2^m$, $c = 1$ and accordingly $n = 2^m - 1$.

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* Miller has shown that, if t does not transform the operators of this H into their inverses, the order of ts^c is the least common multiple of the orders of t and s^c. These Transactions, vol. 4 (1903), p. 154.

The order r of the extending operator t divides $\phi(m_1)$ and no operator (other than the identity) is transformed into itself. Evidently ts (where s is any operator of H) effects the same automorphism of H as does t and is of order r. This as well as the following points can be proved by methods exactly like those in the proof of Theorem 1: the orders of the operators $(t^a s)$ outside H divide r if $n^a \not\equiv 1$, mod p, or divide p^m, if $n^a \equiv 1$, mod p; if s is chosen of order p^{m_1}, then the order of $t^a s$ may be p^{m_1} ($n^a \equiv 1$, mod p); that the cyclic subgroup of order p^{m_1} generated by such a $t^a s$ is not an invariant subgroup of G. Hence, the cyclic subgroups of order p^{m_1} in H which are the only ones of this order invariant under every operator of G, are characteristic in G (in fact they form a characteristic set*). Moreover, since no operator outside of H is commutative with each operator of a cyclic subgroup of order p^{m_1} in H, therefore H is the direct product of the only cyclic subgroups of order p^{m_1} that are invariant in G and the cyclic subgroups of G whose orders are powers of p and whose operators are commutative with the individual operators of the said cyclic subgroups of order p^{m_1}. Thus H is characteristic in G, and the remainder of the proof proceeds as in the preceding theorem.

6. From Theorem 2 a more general proposition in which H is an abelian group of odd order is easily deduced. The requirement that $n \not\equiv 1$, mod p, can be made by stating that no operator (excepting the identity) is transformed into itself by the extending operator. It will now be shown that *if a group G is formed by extending an abelian group H of odd order by an operator from its group of isomorphisms which transforms every one of its operators into the same power of itself, not the first power, then the I of G is the holomorph of H and is a complete group.*

Let $h = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$, where $p_1, \cdots, p_k$ are distinct odd primes and $p_1 > p_2 > \cdots > p_k$. We may take $k > 1$ since Theorem 2 covers the case for $k = 1$. Let $H_1, \cdots, H_k$ be the abelian subgroups of orders $p_1^{m_1}, \cdots, p_k^{m_k}$, respectively.† If $p_1^{w_1}, \cdots, p_k^{w_k}$ are the respective orders of the cyclic subgroups of largest orders in $H_1, \cdots, H_k$, then the order r of the extending operator t is the least common multiple of the exponents to which n appertains with respect to the moduli $p_1^{w_1}, \cdots, p_k^{w_k}$, n being the power into which t transforms each operator of H. Let s_i be an operator of order p^{w_i} in H_i and let t be made up of the commutative cycles $t_1, \cdots, t_k$ where the order of t_i is r_i and where t_i transforms s_i into its nth power but is commutative with all the other s's ($i = 1, \cdots, k$); then $t = t_1 \cdot \cdots \cdot t_k$.

Evidently ts (where s is any operator of H) is of order r, and the cycle $t_j s_j$ (where s_j is any operator of H_j, $j = 1, \cdots, k$) will always be of order r_j and r is the least common multiple of the r's; and ts transforms H exactly

*American Journal of Mathematics, vol. 38 (1916), p. 21.

† Burnside, loc. cit., § 38.

as t does. It will next be shown that H is characteristic in G. First, H_1 is characteristic in G since the cyclic subgroup generated by s_1 is invariant under G while any cyclic subgroup of order $p_1^{w_1}$ in G outside H would, since p_1 is the largest prime, necessitate that the cycle $t_1^a s_1$ be of order $p_1^{w_1}$ $(0 < a < r)$. According to the proof under Theorem 1 this cyclic subgroup generated by $t_1^a s_1$ would not be invariant in G; hence with argument similar to that in the proof of Theorem 2, H_1 itself is characteristic in G. Next, H_2 is characteristic in G, since the cyclic subgroup generated by s_2 is invariant under G and each of its operators is commutative with each of the operators of H_1. Any cyclic subgroup of order $p_2^{w_2}$ in G and outside of H would not be invariant, for by the proof of Theorem 1, if the cycle $t_2^a s_2$ generated a cyclic subgroup of order $p_2^{w_2}$, this subgroup would not be invariant in G; also, if some power of $t_1 s_1' \cdot t_2 s_2'$ (where s_1' is an operator of H_1 and s_2' is an operator of H_2) generated a cyclic subgroup of order $p_2^{w_2}$, evidently its operators would not be individually commutative with the operators of H_1 (since in such a power the first cycle would not reduce to the identity). The only operators (of G) whose orders divide $p_2^{w_2}$ and which are commutative with the individual operators of H_1 and those of the cyclic subgroup generated by s_2, are operators of H_2. Hence, H_2 is characteristic in G. Similarly, for $H_3, \cdots, H_k$; and hence the proposition as stated. Or, according to the observation made in the introduction that requiring the particular extending operator to be " from the group of isomorphisms of H " is, here, only another way of saying that the order of the extending operator must be the same as that of the automorphism which it effects, the proposition may be stated as follows:

THEOREM 3. *If a group G is formed by extending an abelian group H of odd order by an operator t which transforms every operator of H into the same power of itself, not the first power, where the order of t equals the order of the automorphism it effects, then the I of G is the holomorph of H and is a complete group.*

From § 2 in which the case for $h = 2p^m$ is discussed, it is apparent that from Theorem 3 follows

Corollary. If a group G is formed by extending an abelian group H whose order $h = 2h'$ where h' is odd and > 1, by an operator t which transforms every operator of H of order > 2 into the same power of itself, not the first power, where the order of t equals the order of the automorphism which it effects, then the I of G is the holomorph of H.

7. If H is an abelian group different from those of the preceding theorems, the following may be stated:

THEOREM 4. *If a group G is formed by extending an abelian group H whose order $h = 2^m h'$ where $m > 1$ and h' is odd and > 1, by an operator t which transforms every operator of H into the same power of itself such that no operator of odd order is transformed into its first power and such that the order of t equals*

the order of the automorphism of H which it effects and (*a*) *the order of the automorphism of the operators of odd order which t effects is divisible by 2^n where H contains an operator of order 2^n but none of order 2^{n+1}, or else* (*b*) *t transforms into its inverse every one of H's operators whose order is a power of 2, then the I of G is the holomorph of H.*

If H' is the abelian subgroup of order h' in H, it can be shown to be characteristic in G by a method analogous to that in Theorem 2. The operators whose orders are powers of 2 in H are the only operators in G which are commutative with the individual operators of H'. Hence H is characteristic in G. The product of t multiplied into any operator of H effects the same automorphism of H as does t, and is of the same order as t. This latter statement about orders will be discussed. If t is commutative with the operators of H whose orders are powers of 2, the statement is evident; if t is not commutative with these operators, the cycle in t which transforms these operators into their same powers is, by § 4, always of an order which divides 2^n. Hence part (*a*) of the theorem is proved. In part (*b*), if t transforms each operator of H into its inverse, G is the generalized dihedral group; if t transforms every operator of H into the same power of itself and this transformation takes operators whose orders are powers of 2 into their inverses but operators of odd orders not into their inverses (and none excepting the identity into itself), then one cycle of ts (where s is any operator of H including the identity) is always of order 2 while the order of the other part is a constant > 2. Accordingly, as in the other theorems, the I of G may be written as a transitive substitution group on h letters and contains an invariant subgroup simply isomorphic with H; etc.

8. One part of the preceding theorem is a special case of the proposition that *if G is formed by extending an abelian group H which is the direct product of two groups H' and H'' whose orders are relatively prime and where the order of H'' is not divisible by 4, by an operator t which is commutative with the operators of H' but which transforms every operator of H'' of order > 2 into the same power of itself, not the first power, and where the order of t equals the order of the automorphism of H'' which it effects and is divisible by the order of each operator of H', then the I of G is the holomorph of H.*

This proposition as well as the preceding theorems and Professor Miller's theorem on the I of a generalized dicyclic group* are special cases of the following theorem, each being a case in which it can be proved that H is characteristic in G, that the I of G contains an invariant subgroup simply isomorphic with H, and that the subgroup H can be automorphic just as any abelian group simply isomorphic with H can.

* Miller, Blichfeldt, and Dickson, loc. cit., p. 170; cf. also the theorem following the one on the dicyclic group.

THEOREM 5. *If G is formed by extending an abelian group H by an operator t which transforms every operator of H into the same power of itself and which is such that ts_i ($i = 1, \cdots, h$ where $s_1, \cdots, s_h$ are the operators of H) are all of the same order and have a single characteristic operator of H for their same first power in H, then the I of G is the holomorph of H provided H is characteristic in G.*

The hypotheses here include the points established in the detail of the proof of Theorem 1, so that the last two paragraphs in § 1 give discussion sufficient for showing the truth of this theorem.

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CONCERNING THE ZEROS OF THE SOLUTIONS OF CERTAIN DIFFERENTIAL EQUATIONS*

BY

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In volume 42 of the Mathematische Annalen Kneser has shown that the solutions of equations of the form $y^{(n)} + qy = 0$ oscillate an infinite number of times, provided that $x^m q > k > 0$ for sufficiently great values of x, where $n \geqq 2m > 0$ and n is even. In case n is odd the solutions either oscillate an infinite number of times or approach zero together with their first n derivatives. In Section I of this paper the oscillation of solutions of certain linear equations of the second order is discussed and some results are obtained which are not included in Kneser's. Linear equations of a certain trinomial form are discussed in Section II, and in Section III Kneser's conclusions are shown to hold when m is restricted merely to being less than n. A condition of a slightly different nature; namely, that the integral $\int_{x_1}^{x} q\,dx$ diverge, is also shown to be sufficient to insure the oscillation of the solutions an infinite number of times, except possibly when n is odd. It is worth while observing that not only must the solutions vanish an infinite number of times, as Kneser says, but they must change sign an infinite number of times. Some of the conclusions of this section are also applicable to certain non-linear equations. Finally, in Section IV, the solutions of systems of linear equations are considered. It is shown as a particular case of a more general result that under specified conditions not all the functions which form a solution can vanish within an interval whose length is less than an assigned value.

I

Consider the equation

$$y'' + py' + qy = 0, \tag{1}$$

where, for $x \geqq x_1$, $|p| \leqq M$, and $q \geqq h > 0$, M and h being constants such that $4h - M^2 > 0$.

If (1) had a solution y that vanished only a limited number of times for $x > x_1$, and if y' were not negative or zero for all values of x greater than a

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certain x_2, we could take a number $c\,(>x_2)$ such that $y'(c)>0$ and consider the solution y_2 of the equation

$$y''-My'+hy=0, \tag{2}$$

whose derivative vanishes for $x=c$. We can choose x_2 sufficiently great to insure that $y_2=0$ for a value $a\,(<c)$ of x greater than the greatest root of y. If y_1 is the solution of (1) such that $y_1(a)=0$, then $y_1'(x)=0$ for $a<x\leqq c$.* There is therefore a solution y_3 of (1) such that $y_3'(c)=0$ and $y_3(x)=0$ for $a\leqq x<c$. We can assume that $y(c)=y_3(c)$. For values of x a little less than c, $y(x)<y_3(x)$. Therefore $y(x)=y_3(x)$ for some value of x between a and c. But this would give a solution, $y-y_3$, of (1) which vanished twice in (a,c). Since this is impossible, we can assume $y'(x)\leqq 0$ for all sufficiently great values of x.

Consider now a solution y_4 of the equation

$$y''+My'+hy=0 \tag{3}$$

such that $y_4'(c)=0$, where c is such that $y'(c)<0$. Now $y_4(b)=0$ for some b greater than c and the solution of (1) that vanishes for $x=b$ has a derivative that vanishes in (c,b).† Hence if y_5 is the solution of (1) such that $y_5'(c)=0$, then $y_5(x)=0$ for some x in (c,b). We could then assume that $y(c)=y_5(c)$. But this would require that $y(x)=y_5(x)$ for some x in (c,b) exclusive of c. This is, however, impossible and we have therefore completed the proof of

THEOREM I. *If in equation* (1) p *and* q *are continuous functions of* x *such that* $|p|\leqq M$ *and* $q\geqq h>0$ *for every* $x\geqq x_1$, *where* M *and* h *are constants such that* $4h-M^2>0$, *then every solution changes sign an infinite number of times.*‡

In his memoir on Linear Differential Equations of the Second Order§ Sturm considered the effect produced upon the zeros of the solutions of (1) by certain changes in the coefficients. These changes however were not of the kind considered in the following theorem since they did not involve a simultaneous increase in p and q.‖

THEOREM II. *If in the equations*

$$y''+py'+qy=0 \tag{1}$$

and

$$y''+p_1y'+q_1y=0 \tag{4}$$

* Annals of Mathematics, vol. 18 (1917), p. 216.

† Annals of Mathematics, loc. cit., Theorem II, p. 216.

‡ Cf. Kneser, Journal für die reine und angewandte Mathematik, vol. 117 (1897), p. 80.

§ Journal de Mathématiques, vol. 1 (1836), p. 106.

‖ Cf. Annals of Mathematics, loc. cit., p. 216.

p, q, p_1, and q_1 are continuous functions of x such that $p_1 \leqq p \leqq 0$, $q_1 \leqq q$, and $q > 0$, for all values of x greater than x_1, and if every solution of (4) *vanishes an infinite number of times for these values of x, then every solution of* (1) *vanishes an infinite number of times.*

Let y_2 be the solution of (4) that vanishes for $x = a > x_1$. Then since by hypothesis y_2 vanishes an infinite number of times for $x > a$, there is a number $b\,(> a)$ such that $y_2'(b) = 0$. If now y is the solution of (1) that vanishes for $x = a$, then $y'(c) = 0$ and $y(c) > 0$, where $a < c \leqq b$.* Now $y''(c) = -q(c)y(c) < 0$. If $y''(x)$ remained negative for all values of x greater than c, y would vanish for some such x. If $y''(x)$ did not remain negative, it would become zero. Let d be the first value of $x > c$ for which $y''(x) = 0$. Then $p(d)y'(d) + q(d)y(d) = 0$. But $p(d) \leqq 0$ by hypothesis and $y'(d) < 0$ since $y'(c) = 0$ and $y''(x) \leqq 0$ in the interval (c, d). Hence this last equation is an impossible one, unless y vanishes in the interval (c, d). Therefore $y(x)$ has one root, and hence an infinite number of roots, greater than a.

The assumption in the theorem that every solution of (4) vanishes an infinite number of times for $x > x_1$ is more restrictive than it need be. It would be sufficient to assume that for these values of x every root of every solution is followed by a root of the derivative of this solution.

It is natural to inquire whether the restriction that $p \leqq 0$ is a necessary one. As a partial answer to the question we observe that some restriction on p is certainly necessary, since no solution of the equation

$$y'' + 3y' + 2y = 0$$

vanishes more than once, although every solution of the equation

$$y'' + y' + y = 0$$

has an infinite number of roots greater than any assigned number.

If we change the dependent variable from y to $\bar{y}$ by means of the relation $y = x^\lambda \bar{y}$, where λ is a constant, we get

$$\bar{y}'' + \left(p + \frac{2\lambda}{x}\right)\bar{y}' + \left(\frac{\lambda^2 - \lambda}{x^2} + \frac{\lambda p}{x} + q\right)y = 0.$$

If now

$$\frac{\lambda^2 - \lambda}{x^2} + \frac{\lambda p}{x} + q \leqq 0 \tag{5}$$

for $x \geqq x_1 > 0$, no $\bar{y}$, and therefore no y, can vanish more than once for these values of x. Various sufficient conditions for the non-vanishing of y for more than a finite number of values of x greater than x_1 can be obtained from (5) by assigning particular values to λ.

*Annals of Mathematics, loc. cit.

II

Consider the equation

$$y^{(n)} + py^{(n-1)} + qy = 0, \tag{6}$$

where p and q are continuous functions of x in the interval considered and q does not vanish. In counting the number of times a function vanishes each zero will be counted a number of times equal to its multiplicity.

If $y^{(n-1)}(x) = 0$ and $y(x) \neq 0$, the sign of $y^{(n)}(x)$ is opposite that of $q(x)y(x)$. Hence $y^{(n-1)}$ cannot vanish twice within any interval within which y does not vanish, and therefore y' cannot vanish n times within any such interval.

If $q > 0$ for $x \geqq x_1$, a zero of y in this interval followed by a sequence of $n - 1$ zeros of y' without a zero of y must be followed by a zero of y in case n is greater than 2. For $y^{(n-1)}$ has just one zero in this interval, say $x = a$, and must therefore be negative for values of x greater than a, since $y^{(n)}(a)$ is negative. Moreover $y^{(n-2)}$ must have a zero which is equal to, or greater than, a, and for all values of x greater than this zero $y^{(n-2)}$ and $y^{(n-1)}$ would both be negative if y did not vanish. But this is impossible. The exception for $n = 2$ is an exception in fact, as may be seen from a consideration of the equation $y'' + 3y' + 2y = 0$. The solution $y = e^{-x} - e^{-2x}$ has the single zero $x = 0$, and $y' = -e^{-x} + 2e^{-2x}$ vanishes for $x = \log 2$.

If $q < 0$ in an interval, a sequence of $n - 1$ zeros of y' without a zero of y cannot be followed by a zero of y. For if this were possible $y^{(n-1)}$ would change from zero to a negative value while $y^{(n)}$ is positive. But there must be an odd number of zeros of y' between successive zeros of y. Hence *if n is even and $q < 0$, there cannot be more than $n - 3$ zeros of y' between successive zeros of y.**

If n is odd and q does not vanish, there cannot be more than $n - 2$ zeros of y' between successive zeros of y. If $q > 0$ for $x \geqq x_1$ and a zero of y in this interval is followed by a sequence of $n - 2$ zeros of y' without a vanishing of y, $y^{(n-2)}$ must change from a positive to a negative value, and therefore $y^{(n-1)}$ must be negative when $y^{(n-2)}$ is negative. But $y^{(n-1)}$ cannot change from a negative value to zero while y is positive since when $y^{(n-1)}$ is zero $y^{(n)}$ is negative. If then y remained positive, $y^{(n-1)}$ and $y^{(n-2)}$ would remain negative. But this is impossible. Hence *if n is odd and $q > 0$ for $x \geqq x_1$, a zero of y in this interval followed by a sequence of $n - 2$ zeros of y' without a zero of y must be followed by a zero of y.*

Suppose now that, for $x \geqq x_1$, p and q are continous functions of x such that $p \geqq 0$, $q \geqq h > 0$, and p is bounded. If for any value of x under considera-

* For the case $n = 2$ see Bôcher, these Transactions, vol. 3 (1902), p. 199. Cf. also Kneser, Journal für reine und angewandte Mathematik, vol. 116 (1896), p. 181; Mathematische Annalen, vol. 42 (1893), p. 413.

tion y and $y^{(n-1)}$ are both positive, $y^{(n)}$ must be negative. Then $y^{(n-1)}$ must decrease as x increases. If y does not vanish as x increases indefinitely and $y^{(n-1)}$ remains positive, it is possible to select values of x as great as we please for which the absolute value of $y^{(n)}$ is as small as we please. Therefore y will be as small as we please for indefinitely great values of x. But y' can vanish only a limited number of times for $x \geqq x_1$. Hence y must approach zero. If on the other hand, $y^{(n-1)}$ is zero for some value of x for which y is positive, it will be negative for slightly greater values of x. But it cannot increase from a negative value to zero if y remains positive since when it is zero $y^{(n)}$ is negative. And if it remains negative, $y^{(n-2)}$ must steadily decrease. If as x increases indefinitely y does not vanish and $y^{(n-2)}$ does not become negative, it is possible to select values of x as great as we please for which the absolute value of $y^{(n-1)}$ is as small as we please. Then since p is bounded it is possible to select values of x as great as we please for which $py^{(n-1)}$ is as small as we please in absolute value and $y^{(n)}$ is positive. This requires that y come as near to zero as we please, and therefore that y approach zero as a limit. If $y^{(n-2)}$ should become negative it would remain negative and y would ultimately vanish. This completes the proof of

THEOREM III. *If in equation* (6), *for* $x \geqq x_1$, p *and* q *are continuous functions of* x *such that* $p \geqq 0$, $q \geqq h > 0$, *and* p *is bounded, every solution either vanishes an infinite number of times or approaches zero as* x *increases indefinitely.*

III

We consider now equations of the form

$$y^{(n)} + qy = 0, \tag{7}$$

where q is a continuous function of x for $x \geqq x_1$. For any number $a \geqq x_1$ and any solution y of (7) we have

$$y(x) = y(a) + y'(a)(x-a) + \cdots + y^{(n-1)}(a)\frac{(x-a)^{n-1}}{(n-1)!} + y^{(n)}(x_0)\frac{(x-a)^n}{n!}, \tag{8}$$

where $x \geqq x_1$, $x_0 = a + \theta(x-a)$, and $0 < \theta < 1$. If now the equation

$$y(a) + y'(a)(x-a) + \cdots + y^{(n-1)}(a)\frac{(x-a)^{n-1}}{(n-1)!} = 0 \tag{9}$$

is satisfied by a positive value of $x - a$, say $b - a$, then $y(x) = 0$ for some x between a and b, provided that $q > 0$. For if this were not the case we could assume $y > 0$ for $a < x \leqq b$, and if we put $x = b$ in (8) the left member would be positive while the right member would be negative by virtue of (7)

and (9). If, on the other hand, $q < 0$ and $y(a) \geqq 0$, the first root b of $y(x) = 0$ that is greater than a must be such that $b - a$ exceeds the least positive value of $x - a$ that satisfies (9). In particular, if no positive value of $x - a$ satisfies (9), y cannot vanish for any value of x greater than a.

We get immediately from (7)

$$y^{(n-1)}(x) = -\int_{x_1}^{x} qydx + y^{(n-1)}(x_1).$$

If now y is greater than, or equal to, zero for $x \geqq x_1$ and $q \geqq 0$, the integral $\int_{x_1}^{x} qydx$ cannot diverge as x increases indefinitely. For if it did, $y^{(n-1)}$ would become negative, and, since $y^{(n)}$ is by hypothesis negative or zero, y would become negative, contrary to the hypothesis.

THEOREM IV. *If $q \geqq 0$ when $x \geqq x_1$ and y is a solution of* (7) *such that the integral $\int_{x_1}^{x} qydx$ diverges as x increases indefinitely, y must change sign an infinite number of times.*

In case $q \leqq 0$ any solution that does not change sign an infinite number of times must become infinite if the integral diverges, since $y^{(n-1)}$ becomes and remains greater than any given positive number.

A direct application of Theorem IV shows that if the integral $\int_{x_1}^{x} qdx$ diverges there can be no solution y which exceeds a given positive number h for $x \geqq x_1$. Moreover if y does not change sign y' cannot vanish more than $n - 1$ times. Hence any solution which does not change sign an infinite number of times must approach zero as x increases indefinitely.

Now we have already made use of the fact that if two consecutive derivatives of y are negative or zero for all values of x greater than a given number a, y must ultimately be negative; and if two consecutive derivatives are positive or zero for these values of x, y must become positively infinite with x. Moreover if we assume that y is not negative for x greater than a, $y^{(n)}$ must be negative or zero for these values of x. Hence if $y^{(n-1)}$ were negative or zero for any such value of x it would be negative for all greater values and y would change sign. We must assume then that $y^{(n-1)}$ is positive when x is greater than a. If $y^{(n-2)}$ were positive or zero for such a value of x, it would be positive for all greater values and y would become infinite with x. We must assume then that $y^{(n-2)}$ is negative when x is greater than a. By continuing this argument we conclude that when n is even y'' must be negative for these values of x. On the other hand, the supposition that y approaches zero requires that y' become negative and hence the supposition that y does not change sign for x greater than a is untenable.

When n is odd this argument leads to the conclusion that if y does not change sign an infinite number of times, y' must become and remain negative. But this is not inconsistent with the fact that y approaches zero as x increases

indefinitely,* as may be seen from the function $y = e^{-x}$, which is a solution of (7) when n is any positive odd integer and $q \equiv 1$. If however y vanishes at all for $x \geqq x_1$ it must change sign an infinite number of times, since otherwise it would approach zero and y' would change sign from positive to negative. This would require y'' to be negative for some value of x greater than x_1. It would then be negative for all greater values of x and y would change sign. This completes the proof of

THEOREM V. *If $q > 0$ and the integral $\int_{x_1}^{x} q dx$ diverges as x increases indefinitely every solution of* (7) *must change sign an infinite number of times for $x \geqq x_1$ in case n is even, and in case n is odd such a solution must either change sign an infinite number of times or not vanish at all for $x \geqq x_1$.*†

Suppose now that we have

$$x^{m_1} q > h > 0$$

for $x \geqq x_1$. This inequality will be strengthened if we replace m_1 by m, where $m > m_1$. We can so select m that $1/(n - m)$ is an integer greater than 1, whatever value m_1 may have less than n. If y did not change sign an infinite number of times or did not approach zero as x increases indefinitely, there would be a positive number h_1 such that $y(x) > h_1$ for sufficiently great values of x. We proceed to prove that this is impossible and that therefore y must either change sign an infinite number of times or approach zero as x increases indefinitely. Moreover it follows from the proof of Theorem V that the latter can occur only under certain circumstances when n is odd.

We take x_1 sufficiently great so that $y(x) > h_1$ for $x \geqq x_1$ and consider the consequences of the inequality

$$y^{(n)}(x) < -\frac{k}{x^m}, \tag{10}$$

where $k = hh_1 > 0$. From (10) we get

$$y^{(n-1)}(x) \leqq \frac{k}{m-1}\left(\frac{1}{x^{m-1}} - \frac{1}{x_1^{m-1}}\right) + y^{(n-1)}(x_1).$$

If

$$y^{(n-1)}(x_1) - \frac{k}{(m + 1)x_1^{m-1}} < 0,$$

$y^{(n-1)}(x)$ would be negative for sufficiently great values of x and y would become negative. We assume therefore that if $n > 1$

$$y^{(n-1)}(x_1) - \frac{k}{(m-1)x_1^{m-1}} \geqq 0.$$

* In this case $y', y'', \cdots, y^{(n-1)}$ all approach zero. Cf. Kneser, loc. cit., vol. 42, p. 435.

† Cf. the result given by Kneser, loc. cit., p. 420.

Moreover if the expression in the left member of this inequality should become negative as x_1 increases, we could apply the preceding argument for a properly chosen x_1. There remains then to be considered the case in which

$$y^{(n-1)}(x) - \frac{k}{(m-1)x^{m-1}} \geqq 0$$

for $x \geqq x_1$. From this inequality we get

$$y^{(n-2)}(x) \geqq \frac{k}{(m-1)(m-2)}\left(\frac{1}{x_1^{m-2}} - \frac{1}{x^{m-2}}\right) + y^{(n-2)}(x_1).$$

If $n > 2$ and $y^{(n-2)}(x)$ is to remain negative as x increases we must have

$$y^{(n-2)}(x) + \frac{k}{(m-1)(m-2)x^{m-2}} \leqq 0.$$

Hence

$$y^{(n-3)}(x) \leqq \frac{k}{(m-1)(m-2)(m-3)}\left(\frac{1}{x^{m-3}} - \frac{1}{x_1^{m-3}}\right) + y^{(n-3)}(x_1).$$

Now if y does not change sign as x increases, no derivative of index $n - 2i - 1$ can become negative if all the derivatives with indices of the form $n - 2j$, where $j \leqq i$, remain negative. There are therefore but two conceivable results of a continuation of this argument:

(a) *The derivatives of y from the nth to the first inclusive are of alternate signs for $x \geqq x_1$.* In case n is odd this supposition leads to the inequality

$$y(x) \leqq \frac{-k}{(m-1)(m-2)\cdots(m-n)}(x_1^{n-m} - x^{n-m}) + y(x_1).$$

From this we conclude that y must become negative.

When n is even the situation is not so simple. In this case we have

$$y'(x) \geqq \frac{k}{(m-1)(m-2)\cdots(m-n+1)x^{m-n+1}}. \tag{11}$$

Hence

$$y(x) \geqq \frac{-k}{(m-1)\cdots(m-n+1)(m-n)}(x^{n-m} - x_1^{n-m}) + y(x_1).$$

There is therefore a positive number l_1 such that $y(x) > l_1 x^{n-m}$ for sufficiently great values of x, and we can replace (10) by

$$y^{(n)}(x) < \frac{-k_1}{x^{2m-n}} \qquad (k_1 > 0).$$

This in turn gives us $y(x) > l_2 x^{2(n-m)}$ $(l_2 > 0)$, and we can accordingly replace (10) by

$$y^{(n)}(x) < -\frac{k_2}{x^{3m-2n}} \qquad (k_2 > 0).$$

A continuation of this process leads finally to the relation

$$y^{(n)}(x) < \frac{-k_r}{x^{n-1}} \qquad (k_r > 0),$$

which in turn gives us

$$y'(x) \leqq \frac{k_r}{(n-2)!} \log \frac{x_1}{x} + y'(x_1).$$

But this is contrary to (11).

(*b*) *The derivative of index* $n - 2i$ *is positive for sufficiently great values of* x, *while all the derivatives with indices of the form* $n - 2j\,(0 \leqq j < i)$ *are negative for* $x \geqq x_1$ *and all the derivatives with indices of the form* $n - 2j + 1\,(1 \leqq j \leqq i)$ *are positive for these values of* x.

Since $y^{(n-2i)}(x)$ is positive and increasing, we should have $y(x) > l_1 x^{n-2i}$ $(l_1 > 0)$ for sufficiently great values of x. We could therefore replace (10) by

$$y^{(n)}(x) < \frac{-k_1}{x^{m-n+2i}} \qquad (k_1 > 0).$$

But a series of steps similar to those described under (*a*) shows that this inequality requires that $y^{(n-2i+1)}(x)$ be negative for sufficiently great values of x. This however contradicts our assumption.

This completes the proof of

THEOREM VI. *If in the equation*

$$y^{(n)} + qy = 0$$

q *is a positive infinitesimal of order less than* n *with respect to* $1/x\,(x \geqq x_1)$, y *must change sign an infinite number of times as* x *increases from* x_1, *in case* n *is even. If* n *is odd* y *must change sign an infinite number of times as* x *increases from* x_1 *or not vanish at all for* $x \geqq x_1$.

The restriction that $m < n$ is essential since such an equation as

$$y^{\text{iv}} + \frac{15y}{16x^4} = 0,$$

for example, has $y = x^{5/2}$ for a solution.

Some of the foregoing results can be extended to certain non-linear equations of the form

$$y^{(n)} + f(x, y) = 0. \tag{12}$$

Suppose that $f(x, y)$ has the following properties:

1. It is continuous for x and y finite and $x \geqq x_1$.

2. When $x \geqq x_1$, it is positive or negative according as y is positive or negative.

3. For any $x\ (\geqq x_1)$ and any two finite values of y, as y_1 and y_2,

$$|f(x, y_1) - f(x, y_2)| < A|y_1 - y_2|,$$

where A is a constant.

No solution has a singular point for any finite value of $x \geqq x_1$. If a solution y does not change sign an infinite number of times as x increases from x_1 it must ultimately either increase monotonically or decrease monotonically, since y' cannot vanish more than $n - 1$ distinct times within an interval within which y does not vanish. But y can approach zero monotonically only in case it does not vanish at all for $x \geqq x_1$ and n is odd. A repetition of the analysis used in the proof of Theorem IV shows that, if y is such that the integral $\int_{x_1}^{x} f(x, y)\,dx$ diverges as x increases without limit, it must change sign an infinite number of times.

IV

In the *Annals of Mathematics* for June, 1917, I have established the existence of a minimum length for an interval within which any solution of a linear homogeneous differential equation of order n and its first $n - 1$ derivatives can all vanish. This raises the question as to the existence of such minimum intervals for the solutions of more general classes of equations. I show here that these minimum intervals do exist for systems of equations of the first order subject to certain restrictions. In fact a somewhat more general result is established from which the existence of the intervals in question can be inferred as a special case.*

Consider the system of equations

$$(A)\qquad \begin{aligned} \frac{dy_1}{dx} &= f_1(x, y_1, y_2, \cdots, y_n), \\ \frac{dy_2}{dx} &= f_2(x, y_1, y_2, \cdots, y_n), \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ \frac{dy_n}{dx} &= f_n(x, y_1, y_2, \cdots, y_n), \end{aligned}$$

where the f's are continuous functions of their arguments when x is in the interval (a, b) and $y_1, y_2, \cdots, y_n$ are finite. Moreover the inequalities

$$|f_i(x, y_1, y_2, \cdots, y_n) - f_i(x, z_1, z_2, \cdots, z_n)| < \sum_{k=1}^{n} L_k|y_k - z_k|,$$
$$(i = 1, 2, \ldots, n)$$

* This more general result was not contained in the paper presented to the Society. It was suggested to me later by Dr. J. F. Ritt.

where $L_1, L_2, \cdots, L_n$ are certain positive constants, are supposed to be satisfied for any finite values of y_k and z_k.

Let $y_1, y_2, \cdots, y_n$ and $z_1, z_2, \cdots, z_n$ be two solutions of these equations such that each of the differences $y_i - z_i$ vanishes somewhere in (a, b). Picard's method of successive approximations, which is applicable to this case, shows that the functions y_i and z_i remain finite within (a, b). The difference $y_i - z_i$ has therefore a maximum absolute value which we shall represent by M_i. Now

$$\left|\frac{d(y_i - z_i)}{dx}\right| = |f_i(x, y_1, y_2, \cdots, y_n) - f_i(x, z_1, z_2, \cdots, z_n)| < \sum_{k=1}^{n} L_k |y_k - z_k|.$$

Moreover $y_i - z_i = 0$ somewhere within (a, b). Hence

$$M_i < \sum_{k=1}^{n} L_k M_k \cdot \rho,$$

where $b - a = \rho$.

If in this inequality we give to i the values $1, 2, \cdots, n$ in turn and multiply the resulting inequalities by $L_1, L_2, \cdots, L_n$ respectively, and add, we get, after dividing by $\sum L_k M_k$,

$$1 < \sum L_k \rho,$$

or

$$\rho > \frac{1}{\sum L_k}.$$

We have therefore

THEOREM VII. *If the corresponding functions of two solutions of equations A are equal somewhere within the interval (a, b), the length of this interval must exceed $1/\sum L_k$.*

If we assume further that the functions f_i are such that

$$f_i(x, 0, 0, \cdots, 0) = 0$$

within (a, b), then $0, 0, \cdots, 0$ is a solution of A and we can apply Theorem VII to this and the solution $y_1, y_2, \cdots, y_n$. This leads to the conclusion that if $\rho \leqq 1/\sum L_k$ *no non-identically vanishing solution* $y_1, y_2, \cdots, y_n$ *of equations A in which* $f_i(x, 0, 0, \cdots, 0) = 0$ *within an interval* (a, b) *of length* ρ *can be such that all the y's vanish within* (a, b). *

This conclusion can not be drawn from the conditions of Theorem VII, as may be seen from a consideration of the system

$$\frac{dy_1}{dx} = \frac{e^x}{3} - y_1 - \tfrac{2}{3}y_2,$$

$$\frac{dy_2}{dx} = -x + \tfrac{4}{3}y_1 + y_2.$$

This system satisfies all the conditions of the theorem for any finite interval, but not the condition that $f_i(x, 0, 0, \cdots, 0) = 0$. On the other hand, it has the solution

$$y_1 = (6e^{1/3} + \tfrac{1}{2}e^{-2/3})e^{x/3} - (12e^{-1/3} + \tfrac{1}{2}e^{-4/3})e^{-x/3} - 6x,$$

$$y_2 = -(12e^{1/3} + e^{-2/3})e^{x/3} + (12e^{-1/3} + \tfrac{1}{2}e^{-4/3})e^{-x/3} + \tfrac{1}{2}e^{x} + 9x + 9$$

and y_1 and y_2 both vanish when $x = -1$.

Analogous results hold in case the f's are analytic functions of the complex arguments $x, y_1, y_2, \cdots, y_n$ when x remains within the circle of center x_0 and radius r (or upon its circumference) and $y_1, y_2, \cdots, y_n$ are finite, subject to the condition that for any fixed k from 1 to n inclusive and every i from 1 to n inclusive

$$\left|\frac{\partial f_i}{\partial y_k}\right| < L_k$$

for these values of the argument.

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DIFFERENTIATION WITH RESPECT TO A FUNCTION OF LIMITED VARIATION*

BY

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Lebesgue, Radon, and Young† have defined integrals with respect to a function of limited variation, and these are generalizations of the Stieltjes integral. The next step which suggests itself is a definition of the corresponding derivative. Such a definition is given in this paper, and the fundamental property of a derivative is proved by means of a modification of Vitali's theorem. The steps taken are parallels of steps given by de la Vallée Poussin‡ in the theory of Lebesgue plane sets. Since this paper was first written, a paper by Young§ has appeared, giving a slightly different definition of the derivative, and an entirely different treatment. The applications at the end of this paper are not given by Young.

Definition of derived numbers and derivative. Consider two functions of x, $F(x)$, $\alpha(x)$, defined in the fundamental interval, $0 \leqq x \leqq 1$. The ratio

$$\frac{F(x+\epsilon) - F(x-\epsilon)}{\alpha(x+\epsilon) - \alpha(x-\epsilon)} = \frac{\Delta F}{\Delta \alpha}$$

may have upper and lower limits as ϵ approaches 0. We define these as the upper and lower derived numbers of $F(x)$ with respect to $\alpha(x)$, and we use the notation

$$\overline{D}_\alpha F(x) = \overline{\lim_{\epsilon \doteq 0}} \frac{\Delta F}{\Delta \alpha}, \qquad \underline{D}_\alpha F(x) = \underline{\lim_{\epsilon \doteq 0}} \frac{\Delta F}{\Delta \alpha}.$$

For x equal to 0 or 1 it is necessary to add a convention whereby $F(x)$, $\alpha(x)$ are continued beyond the range $(0, 1)$, and have values equal to their values at 0 or 1 respectively. If the two derived numbers are finite and equal,

* Presented to the Society, September 6, 1918.

† Lebesgue, *Comptes Rendus*, vol. 150 (1910), p. 86; Radon, *Wiener Sitzungsberichte*, vol. 122, section 2a (1913), p. 1295; W. H. Young, *Proceedings of the London Mathematical Society*, vol. 13 (1914), p. 109.

‡ De la Vallée Poussin, these *Transactions*, vol. 16 (1915), p. 435.

§ W. H. Young, *Proceedings of the London Mathematical Society*, vol. 15 (1916), p. 35.

that is, if the relative change ratio $\Delta F/\Delta\alpha$ possesses a finite limit, this limit is called the α-derivative of $F(x)$, and is denoted by

$$D_\alpha F(x).$$

An interval $(x-\epsilon, x+\epsilon)$ we shall call a *central interval* with center at x.

If $\alpha(x)$, $F(x)$, are functions of limited variation, we can define absolutely additive functions of sets measurable Borel

$$\int_E d\alpha, \quad \int_E dF,$$

and also the corresponding modular integrals, denoted according to Radon by

$$\int_E |d\alpha|, \quad \int_E |dF|,$$

or according to the writer by

$$\int_E d\omega, \quad \int_E d\Omega,$$

where $\omega(x)$, $\Omega(x)$, are the variation functions corresponding to $\alpha(x)$, $F(x)$. These modular integrals are additive, finite, and non-negative.

Absolute continuity. A function $F(x)$ is said to be absolutely continuous relative to $\alpha(x)$, a function of limited variation, if given any positive ϵ we can find δ so that

$$\int_e d\Omega(x) < \epsilon,$$

for all sets e measurable Borel such that

$$\int_e d\omega(x) < \delta,$$

where $\omega(x)$, $\Omega(x)$ are the variation functions corresponding to $\alpha(x)$, $F(x)$. We desire to prove the fundamental proposition:

THEOREM. *If $F(x)$ is absolutely continuous relative to $\alpha(x)$, it possesses a finite α-derivative nearly everywhere (ω) [that is, except for a point set e for which $\int_e d\omega(x) = 0$]; this α-derivative is summable (α) where it exists, and if E is any set measurable Borel,*

$$\int_E dF(x) = \int_E D_\alpha F(x)\, d\alpha(x).$$

In this, $D_\alpha F(x)$ denotes the α-derivative of $F(x)$ where it exists, and any finite value where it does not.

Before we can prove this proposition, it is necessary to prove two lemmas.

LEMMA 1. *Given any positive* ϵ *and a non-decreasing function* $\omega(x)$, *any set* E *measurable Borel can be enclosed strictly in a finite or denumerable system* A *of disjoint intervals (that is, intervals with no points common to any two) in such a way that*

$$\int_A d\omega < \int_E d\omega + \epsilon.$$

Let D be the set of points at which $\omega(x)$ has finite discontinuities; then D consists at most of a denumerable set of points. Resolve $\omega(x)$ into $\omega_1(x)$, a continuous non-decreasing function, and $\omega_2(x)$, a non-decreasing function which is stationary except when x passes a point of D. The integral $\int_E d\omega_1(x)$ will be a continuous additive non-negative function of sets. Hence,* given any positive ϵ, we can enclose E strictly in a finite or denumerable system A_1 of disjoint intervals so that

$$\int_{A_1} d\omega_1(x) < \int_E d\omega_1(x) + \tfrac{1}{2}\epsilon.$$

The set $CE \cdot D$ consists at most of a denumerable set of points D'. Then, since

$$\int_{D'} d\omega(x) = \int_{D'} d\omega_2(x)$$

is a convergent series of positive terms (considered as the sum of the discontinuities of $\omega(x)$ at the denumerable set of points D'), we can choose a finite n and the finite set of points D_n so that

$$\int_{D'} d\omega_2(x) < \int_{D_n} d\omega_2(x) + \tfrac{1}{2}\epsilon.$$

From A_1 cut out the points belonging to D_n, which are finite in number. Then $A = A_1 \cdot CD_n$ still forms a denumerable system of intervals enclosing E strictly; for the points D_n belong to CE. The set A is the same as A_1 except for the exclusion of a finite set of points, or

$$\int_A d\omega_1(x) = \int_{A_1} d\omega_1(x) < \int_E d\omega_1(x) + \tfrac{1}{2}\epsilon.$$

Again, from the way in which D_n was chosen,

$$\int_{CE\cdot D\cdot CD_n} d\omega_2(x) < \tfrac{1}{2}\epsilon.$$

But

$$\int_e d\omega_2(x) = 0$$

* De la Vallée Poussin, loc. cit., p. 470.

over any set e not containing points of D. Hence

$$\int_{CE \cdot CD_n} d\omega_2(x) < \tfrac{1}{2}\epsilon, \qquad \int_{A_1 \cdot CE \cdot CD_n} d\omega_2(x) < \tfrac{1}{2}\epsilon,$$

and, since $A = A_1 \cdot CD_n$,

$$\int_{A \cdot CE} d\omega_2(x) < \tfrac{1}{2}\epsilon, \qquad \int_A d\omega_2(x) < \int_E d\omega_2(x) + \tfrac{1}{2}\epsilon.$$

Hence

$$\int_A d\omega(x) = \int_A d\omega_1(x) + \int_A d\omega_2(x) < \int_E d\omega(x) + \epsilon.$$

The lemma is proved.

The following lemma is a generalization of Vitali's theorem.*

LEMMA 2. *Given a set E measurable Borel, and an infinite family Γ of central intervals, such that each point of E is the center of an infinity of central intervals as small as we please; then a set B can be found consisting of a finite or denumerable number of disjoint intervals chosen from Γ, such that B covers nearly all E (that is, except for a point set of ω-measure 0) and such that the ω-measure of B differs from that of E by as little as we please.*

That is to say, given any positive ϵ, we can find B so that

$$\int_{E \cdot CB} d\omega(x) = 0, \qquad \int_B d\omega(x) < \int_E d\omega(x) + \epsilon.$$

By means of Lemma 1, given any positive ϵ, we can enclose E strictly in a denumerable set of disjoint intervals A, so that

$$\int_A d\omega' < \int_E d\omega' + \epsilon,$$

where $\omega'(x) = \omega(x) + x$. In what follows we denote the ω'-measure of a set E simply by mE. Then mE is the sum of the ω-measure and the usual Lebesgue measure. The intervals A are not necessarily central intervals. From the family Γ eliminate the intervals which have points in common with CA. The remaining family Γ_1 will possess the same property relative to E, for each point of E is interior (strictly) to one or other of the intervals A.

We affirm that with a finite number of disjoint intervals of Γ_1 we can cover a part e_1 of E such that $me_1 > kmE$, where k is any number less than one-third. For† let the set CE be enclosed strictly in a denumerable set D of open intervals so that mD is arbitrarily close to mCE. Then CD is a closed set contained in E, and given any positive ϵ we can make $mCD > mE - \epsilon$. By the Heine-

* See de la Vallée Poussin, *Cours d'Analyse*, second edition, vol. 2, p. 110.

† This part of the proof is due to the Editors of these Transactions.

Borel Theorem* CD can be covered by a finite number of intervals E_n chosen from the family Γ_1. Then

$$mE_n \geqq mCD > mE - \epsilon.^*$$

From these intervals choose first that one having the greatest ω'-measure, and eliminate those which have any point in common with it. Next choose the remaining interval which has the greatest ω'-measure, and so on. After a finite number of such steps we shall have chosen a finite number of intervals, and the process will terminate. Each time intervals are eliminated, the ω'-measure of the interval retained will be at least one-third of the ω'-measure of the interval covered by it and all intervals eliminated as overlapping it. Hence the measure of all intervals retained will be at least $\frac{1}{3}mE_n$, or will be greater than $\frac{1}{3}mE - \epsilon$. The part of E not covered, that is $E - e_1$, is therefore of measure less than $mA - \frac{1}{3}mE + \epsilon$, or less than $\frac{2}{3}mE + 2\epsilon$. Hence

$$me_1 > \tfrac{1}{3}mE - 2\epsilon.$$

Our affirmation is proved, that with a finite number of disjoint intervals belonging to Γ_1 we can cover a part e_1 of E so that

$$me_1 > kmE,$$

where k is any number less than one-third.

After we have thus chosen e_1, omit from Γ_1 those intervals which overlap e_1, and let Γ_2 be the remainder. Then Γ_2 will have the same properties relative to $E - e_1$ as Γ_1 has relative to E. We can by the same process cover a portion e_2 of $E - e_1$, such that

$$me_2 > km(E - e_1),$$

by a finite number of intervals of Γ_1. Continuing the process, we obtain a system B of disjoint intervals of the family Γ, which are at most denumerable. Moreover the set $B \cdot E = \sum e_n$. But

$$me_n > k(mE - me_1 - \cdots - me_{n-1}),$$

and the term in the parenthesis is non-negative, and therefore $\sum me_n$ is convergent. Thus me_n approaches zero, or

$$m(\sum e_n) = \sum me_n = mE.$$

Then

$$mE \cdot B = mE, \qquad mE \cdot CB = 0, \qquad \int_{E \cdot CB} d\omega' = 0.$$

But $\omega'(x) = \omega(x) + x$, so that

* Cf. de la Vallée Poussin, *Intégrales de Lebesgue*, pp. 13–15.

$$\int_{E\cdot CB} d\omega \leqq \int_{E\cdot CB} d\omega' = 0;$$

and, since

$$\int_{A-E} d\omega \leqq \int_{A-E} d\omega' < \epsilon, \qquad \int_A d\omega < \int_E d\omega + \epsilon.$$

But B is contained in A, or

$$\int_B d\omega < \int_E d\omega + \epsilon.$$

The lemma is proved.

To return to our original proposition, let us prove it first in the case where $\alpha(x)$ is a non-decreasing function, so that $\alpha(x) = \omega(x)$.

In any set e, if $\bar{D}_\omega F(x) \geqq l$ (this inequality being considered to hold if $\bar{D}_\omega F(x) = +\infty$), we shall show that

$$\int_e dF \geqq l \int_e d\omega.$$

When this is proved, it will follow as a corollary, and can also be established directly by parallel reasoning, that a similar conclusion holds if the signs of inequality are reversed, or if $\bar{D}$ is replaced by $\underline{D}$, or both. At any rate, it is sufficient to give the proof for the case first mentioned. Corresponding to every point in e (which is measurable Borel) we can find an infinity of central intervals as small as we please, such that for each

$$\frac{\Delta F}{\Delta \omega} > l - \epsilon',$$

given any positive ϵ'. These form a family Γ having the Vitali property relative to e. Since $F(x)$ is absolutely continuous with respect to $\omega(x)$, given any positive ϵ we can find δ so that

$$\int_e d\Omega < \epsilon,$$

for all sets e for which

$$\int_e d\omega < \delta.$$

Using Lemma 2, we can define a set B consisting of a denumerable system of disjoint intervals belonging to Γ, such that

$$\int_{e\cdot CB} d\omega = 0,$$

whence

$$\int_{e\cdot CB} d\Omega = 0.$$

and such that

$$\int_{B \cdot Ce} d\omega < \delta,$$

whence

$$\int_{B \cdot Ce} d\Omega < \epsilon.$$

Then

$$\left| \int_e d\omega - \int_B d\omega \right| < \delta, \qquad \left| \int_e dF - \int_B dF \right| < \epsilon.$$

But for each of the intervals B,

$$\frac{\Delta F}{\Delta \omega} > l - \epsilon', \qquad \Delta F > l\Delta\omega - \epsilon'\Delta\omega, \qquad \int_B dF > l \int_B d\omega - \epsilon' \int_B d\omega,$$

$$\int_e dF > (l - \epsilon') \int_e d\omega - \epsilon - |l|\delta - \epsilon'\delta.$$

In the limit, as ϵ, ϵ' approach 0, δ also approaches 0, and

$$\int_e dF \geqq l \int_e d\omega.$$

Since $\int_e dF$ is limited, the ω-measure of e decreases to the limit 0 as l increases indefinitely. Also, since F is absolutely continuous with respect to ω, $\int_e dF$ will also approach the limit 0. Thus the set of points for which $\overline{D}_\omega F(x) = +\infty$ is of ω-measure 0. A similar proof shows that the set for which $\underline{D}_\omega F(x) = -\infty$ is of ω-measure 0.

Take two finite numbers m, M, and divide the interval between them into sub-intervals by $m = l_0 < l_1 < \cdots < l_n = M$, where $\max |l_i - l_{i-1}| <$ a given positive η. Let e_i be the set in E for which $l_{i-1} \leqq \overline{D}_\omega F < l_i$; then

$$l_{i-1} \int_{e_i} d\omega \leqq \int_{e_i} dF \leqq l_i \int_{e_i} d\omega,$$

$$\left| \int_{e_i} dF - l_i \int_{e_i} d\omega \right| \leqq \eta \int_{e_i} d\omega.$$

By summing up for the sets e_i, if E' is the set of points where $m \leqq \overline{D}_\omega F < M$,

$$\left| \int_{E'} dF - \sum_i l_i \int_{e_i} d\omega \right| \leqq \eta \int_{E'} d\omega;$$

in the limit, as η approaches 0, $\overline{D}_\omega F$ is summable (ω) in E', and

$$\int_{E'} dF = \int_{E'} \overline{D}_\omega F d\omega.$$

It has been proved already that $\int_{E-E'} dF$ approaches 0 as M increases and m decreases indefinitely, whence $\overline{D}_\omega F$ is summable (ω) in E, and

$$\int_E dF = \int_E \overline{D}_\omega F d\omega .$$

Similarly it can be proved that $\underline{D}_\omega F$ is summable (ω) in E, and

$$\int_E dF = \int_E \underline{D}_\omega F d\omega .$$

But $\underline{D}_\omega F \leqq \overline{D}_\omega F$, or at every point of E except on a point set of ω-measure 0,

$$\underline{D}_\omega F = \overline{D}_\omega F = D_\omega F ,$$

and $F(x)$ has a finite derivative with respect to $\omega(x)$ nearly everywhere (ω) in E. Also

$$\int_E dF = \int_E D_\omega F d\omega .$$

More generally, if $\alpha(x)$ is a function of limited variation, it is absolutely continuous with respect to its variation function $\omega(x)$. We may split any set E measurable Borel into two subsets E_1, E_2, so that if e is a variable set,

$$\int_{e\cdot E_1} d\alpha = \int_{e\cdot E_1} d\omega ,$$

a non-negative function of sets, and

$$\int_{e\cdot E_2} d\alpha = -\int_{e\cdot E_2} d\omega .$$

We have already proved that

$$\int_{E_1} dF = \int_{E_1} D_\omega F d\omega = \int_{E_1} D_\omega F d\alpha ,$$

and by the same proof $D_\omega \alpha$ exists and is finite nearly everywhere (ω), and

$$\int_{e\cdot E_1} d\omega = \int_{e\cdot E_1} d\alpha = \int_{e\cdot E_1} D_\omega \alpha d\omega ;$$

whence $D_\omega \alpha = \lim \Delta\alpha/\Delta\omega$ exists and equals 1 nearly everywhere (ω) in E_1. It follows that $D_\alpha F$ exists and equals $D_\omega F$ nearly everywhere (ω) in E_1; or

$$\int_{E_1} dF = \int_{E_1} D_\alpha F d\alpha .$$

Similarly in E_2, $D_\alpha F$ exists and equals $-D_\omega F$ nearly everywhere (ω), and

$$\int_{E_2} dF = \int_{E_2} D_\omega F d\omega = -\int_{E_2} D_\omega F d\alpha = \int_{E_2} D_\alpha F d\alpha .$$

By combining E_1 and E_2, it is proved that $D_\alpha F$ exists and is finite in E except for a set of ω-measure 0, that it is summable in E, and that

$$\int_E dF(x) = \int_E D_\alpha F(x) d\alpha(x).$$

Applications. *Reduction of general integral to integral of positive type.* Let $g(x)$ denote the function

$$g(x) = D_\omega \alpha(x),$$

where this derivative $= +1$ or -1, and

$$g(x) = 0$$

otherwise, that is to say on a set of ω-measure 0. On the set E_1, $g(x) = 1$ nearly everywhere (ω), whence, if $f(x)$ is any function summable (ω),

$$\int_{E_1} f(x) d\alpha(x) = \int_{E_1} f(x) d\omega(x) = \int_{E_1} f(x) g(x) d\omega(x).$$

Similarly,

$$\int_{E_2} f(x) d\alpha(x) = -\int_{E_2} f(x) d\omega(x) = \int_{E_2} f(x) g(x) d\omega(x),$$

whence

$$\int_E f(x) d\alpha(x) = \int_E f(x) g(x) d\omega(x) .$$

This proves that any integral with respect to a function of bounded variation $\alpha(x)$ can be expressed as a single integral of positive type, that is to say, an integral with respect to a non-decreasing function $\omega(x)$.

Integration by Parts. Let $F(x) = \alpha^2(x)$, where $\alpha(x)$ is a function of limited variation. The latter function may be expressed as the difference of two that are non-decreasing,

$$\alpha(x) = \beta_1(x) - \beta_2(x),$$

whence

$$F(x) = \alpha^2(x) = \beta_1^2(x) + \beta_2^2(x) - 2\beta_1(x)\beta_2(x),$$
$$\Omega(x) = \beta_1^2(x) + \beta_2^2(x) + 2\beta_1(x)\beta_2(x) = \omega^2(x).$$

The function $\omega(x)$ is limited and less than some finite number K, so that for any interval

$$\Delta\Omega < 2K\,\Delta\omega\,;$$

whence $F(x) = \alpha^2(x)$ is absolutely continuous with respect to $\alpha(x)$. In this case

$$\frac{\Delta F}{\Delta \alpha} = \frac{\alpha^2(x+\epsilon) - \alpha^2(x-\epsilon)}{\alpha(x+\epsilon) - \alpha(x-\epsilon)} = \alpha(x+\epsilon) + \alpha(x-\epsilon).$$

Let $\bar{\alpha}(x)$ denote

$$\lim_{\epsilon \doteq 0} \tfrac{1}{2}[\alpha(x+\epsilon) + \alpha(x-\epsilon)];$$

then

$$D_\alpha \alpha^2(x) = 2\bar{\alpha}(x).$$

By the fundamental proposition,

$$\int_E d\alpha^2(x) = \int_E 2\bar{\alpha}(x)\, d\alpha(x).$$

Let $\alpha_1(x)$, $\alpha_2(x)$ be two functions of limited variation; then

$$\begin{aligned}\int d[(\alpha_1 + \alpha_2)^2] &= \int d\alpha_1^2 + \int d\alpha_2^2 + 2\int d(\alpha_1 \alpha_2) \\ &= 2\int (\bar{\alpha}_1 + \bar{\alpha}_2)\, d(\alpha_1 + \alpha_2) \\ &= 2\int \bar{\alpha}_1\, d\alpha_1 + 2\int \alpha_2\, d\alpha_2 + 2\int \bar{\alpha}_1\, d\alpha_2 + 2\int \bar{\alpha}_2\, d\alpha_1.\end{aligned}$$

So

$$\int_E d(\alpha_1 \alpha_2) = \int_E \bar{\alpha}_1\, d\alpha_2 + \int_E \bar{\alpha}_2\, d\alpha_1.$$

Transposed and written more fully, the last relation becomes

$$\int_E \bar{\alpha}_1(x)\, d\alpha_2(x) = \int_E d[\alpha_1(x)\, \alpha_2(x)] - \int_E \bar{\alpha}_2(x)\, d\alpha_1(x).$$

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LINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH A BOUNDARY CONDITION*

BY

MINFU TAH HU

CONTENTS

1. Introduction and Notations

It is a well-known fact that linear integral equations of the first and second kinds may be regarded as the limiting cases, as n becomes infinite, of systems of n linear algebraic equations in n variables.

The same idea of passing to a limit suggests that one treat the integro-differential equation

$$(A)\quad \frac{\partial u(x,s)}{\partial x} + \phi(x,s)u(x,s) + \int_a^\beta \psi\begin{pmatrix} s \\ x\,t \end{pmatrix} u(x,t)\,dt = \lambda(x,s)$$

as the limit of a system of n linear differential equations of the first order of the form†

* Presented to the Society, December 28, 1917. The problem treated in this paper was first suggested to me by Professor W. A. Hurwitz, to whom, and to Professor M. Bôcher, I tender my grateful acknowledgment for constant help, suggestions, and criticisms.

† For the system (a) when all the equations are homogeneous, a different integro-differential equation was obtained by Schlesinger (Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 24 (1915), p. 84) by means of a process involving certain changes of the form of the equations (a) before passing to the limit. The equation thereby obtained differs from (A) in that the variable x is complex and all functions involved are analytic functions in x, that the functions u and λ contain another variable r of the same class as s, and that

$$\phi(x,s) \equiv 0 \qquad \text{and} \qquad \lambda(x,s) \equiv \psi\begin{pmatrix} s \\ x\,r \end{pmatrix}.$$

$$
(a)\qquad \begin{aligned}
&\frac{du_1(x)}{dx} + l_{11}(x)u_1(x) + \cdots + l_{1n}(x)u_n(x) = \lambda_1(x),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\frac{du_n(x)}{dx} + l_{n1}(x)u_1(x) + \cdots + l_{nn}(x)u_n(x) = \lambda_n(x).
\end{aligned}
$$

We shall have occasion to adjoin to (A) a boundary condition of the type

$$
(B)\qquad \begin{aligned}
\alpha(s)u(a, s) + \beta(s)u(b, s) + \int_\alpha^\beta [A(s, r)u(a, r)&\\
+ B(s, r)u(b, r)]dr &= \gamma(s).
\end{aligned}
$$

This we shall call a two-point boundary condition since it involves the two values a and b of the variable x. This is obviously the limiting form of the system of linear boundary conditions usually attached to the finite differential system (a), as we let the number of equations increase indefinitely.

Throughout this paper, all variables entering will be real. These variables may be conveniently divided into two classes corresponding respectively to the first and second arguments of the unknown function u in the equations (A), (B). The first class of variables is denoted by such letters as $x, y, z, \xi, \eta, \zeta$, and they take on the values in the closed interval

$$
I: \qquad a \leqq x \leqq b.
$$

We shall speak of this in the future simply as the interval I_x, the subscript x indicating the variable referred to.

The second class of variables is usually denoted by the letters $s, t, r, \sigma, \tau, \rho$, which take on the values in the interval

$$
J: \qquad \alpha \leqq s \leqq \beta.
$$

In the case of functions of two or more variables, each of which is confined to one of the intervals I and J, we interpret the different variables as rectangular coördinates. For instance, the unknown function $u(x, s)$ will be supposed to be defined in the rectangle

$$
I_x J_s: \qquad a \leqq x \leqq b, \qquad \alpha \leqq s \leqq \beta.
$$

In case the variables belong to the same class, we shall have square regions I_{xy} or J_{st}. Likewise, for functions involving more than two variables we have

Then Schlesinger considered also the associated *homogeneous* equation of the type (A) whose solutions are made dependent on the solutions of the former equation. These equations were also treated in a similar fashion in the notation of general analysis by T. H. Hildebrandt (these Transactions, vol. 18 (1917), p. 73). [After the manuscript of the present paper was in the hands of the editors of the Transactions, I was informed by them that a second paper by Hildebrandt was to appear shortly in the Transactions. See vol. 19 (1918), p. 97.]

such regions as $I_{xy} J_s$, $I_{xy} J_{st}$, etc. All these intervals and regions will be understood to be closed.

To simplify our work, we shall assume, unless otherwise stated, that all functions considered are real and continuous (and therefore bounded) in the respective regions in which they are defined. By a *solution* of the equations (A), (B), or any other equation under consideration, we understand, without further specification, a continuous function. A solution of the equations (A), (B), possesses a continuous first derivative with respect to its first argument. A solution which is identically zero will be termed a *trivial solution.*

2. The Integro-Differential Equation

The integro-differential equation

$$(A) \quad \frac{\partial u(x, s)}{\partial x} + \phi(x, s) u(x, s) + \int_a^\beta \psi\begin{pmatrix} s \\ x\,t \end{pmatrix} u(x, t)\, dt = \lambda(x, s)$$

may be reduced, by means of the transformation*

$$u(x, s) = e^{-\int_y^x \varphi(\xi, s) d\xi} v(x, s),$$

where y is regarded as a fixed point in I_x, to the equation

$$(1) \quad \frac{\partial v(x, s)}{\partial x} + \int_a^\beta \psi\begin{pmatrix} s \\ x\,t \end{pmatrix} e^{-\int_y^x [\varphi(\xi, t) - \varphi(\xi, s)] d\xi} \times v(x, t)\, dt = e^{\int_y^x \varphi(\xi, s) d\xi} \lambda(x, s).$$

This equation is the special case of (A) in which the second term of the first member is lacking. Let us write for convenience

$$(2) \quad R\begin{pmatrix} x\,s \\ y \end{pmatrix} = e^{-\int_y^x \varphi(\xi, s) d\xi},$$

so that

$$(3) \quad R\begin{pmatrix} x\,s \\ y \end{pmatrix} R\begin{pmatrix} y\,s \\ \xi \end{pmatrix} = R\begin{pmatrix} x\,s \\ \xi \end{pmatrix}.$$

Changing x in (1) into ξ and integrating from y to x, we find

$$v(x, s) = v(y, s) + \int_y^x R\begin{pmatrix} y\,s \\ \xi \end{pmatrix} \lambda(\xi, s)\, d\xi - \int_y^x \int_a^\beta \psi\begin{pmatrix} s \\ \xi\,t \end{pmatrix} R\begin{pmatrix} y\,s \\ \xi \end{pmatrix} R\begin{pmatrix} \xi\,t \\ y \end{pmatrix} v(\xi, t)\, dt\, d\xi.$$

* This was pointed out to me by Professor Birkhoff.

We may now transform back to $u(x,s)$, getting

$$(A') \qquad \begin{aligned} u(x,s) = R\begin{pmatrix} x\,s \\ y \end{pmatrix} u(y,s) + \int_y^x R\begin{pmatrix} x\,s \\ \xi \end{pmatrix} \lambda(\xi,s)\,d\xi \\ + \int_y^x \int_\alpha^\beta \left[-R\begin{pmatrix} x\,s \\ \xi \end{pmatrix} \psi\begin{pmatrix} s \\ \xi\,t \end{pmatrix} \right] u(\xi,t)\,dt\,d\xi. \end{aligned}$$

This is a special case of the equation

$$(4) \qquad u(x,s) = f(x,s) + \int_y^x \int_\alpha^\beta \theta\begin{pmatrix} x\,s \\ \xi\,t \end{pmatrix} u(\xi,t)\,dt\,d\xi.$$

Let us then consider (4).

The function $\theta\left(\begin{smallmatrix} x\,s \\ \xi\,t \end{smallmatrix}\right)$ (called the kernel of the equation) will be supposed to be continuous in $I_{x\xi} J_{st}$. In its appearance, the equation is intermediate between the Volterra and the Fredholm types; but it behaves like an equation of the Volterra type because of the variable limit of the first integral. Since Volterra's method may be applied almost word for word,* we shall give here only the results.

We are led by the method of successive substitutions to the consideration of the series

$$(5) \qquad \theta_1\begin{pmatrix} x\,s \\ \xi\,t \end{pmatrix} + \theta_2\begin{pmatrix} x\,s \\ \xi\,t \end{pmatrix} + \theta_3\begin{pmatrix} x\,s \\ \xi\,t \end{pmatrix} + \cdots,$$

where

$$\theta_1\begin{pmatrix} x\,s \\ \xi\,t \end{pmatrix} = \theta\begin{pmatrix} x\,s \\ \xi\,t \end{pmatrix},$$

$$\theta_n\begin{pmatrix} x\,s \\ \xi\,t \end{pmatrix} = \int_\xi^x \int_\alpha^\beta \theta_{n-1}\begin{pmatrix} x\,s \\ \eta\,r \end{pmatrix} \theta_1\begin{pmatrix} \eta\,r \\ \xi\,t \end{pmatrix} dr\,d\eta.$$

The series (5) converges absolutely and uniformly in $I_{x\xi} J_{st}$, thus representing a bounded continuous function, $\Theta\left(\begin{smallmatrix} x\,s \\ \xi\,t \end{smallmatrix}\right)$, which shall be called the resolvent function of the kernel $\theta\left(\begin{smallmatrix} x\,s \\ \xi\,t \end{smallmatrix}\right)$.

The kernel and the resolvent function satisfy the resolvent formulæ

$$(6) \qquad \Theta\begin{pmatrix} x\,s \\ \xi\,t \end{pmatrix} = \theta\begin{pmatrix} x\,s \\ \xi\,t \end{pmatrix} + \int_\xi^x \int_\alpha^\beta \Theta\begin{pmatrix} x\,s \\ \eta\,r \end{pmatrix} \theta\begin{pmatrix} \eta\,r \\ \xi\,t \end{pmatrix} dr\,d\eta,$$

$$(7) \qquad \Theta\begin{pmatrix} x\,s \\ \xi\,t \end{pmatrix} = \theta\begin{pmatrix} x\,s \\ \xi\,t \end{pmatrix} + \int_\xi^x \int_\alpha^\beta \theta\begin{pmatrix} x\,s \\ \eta\,r \end{pmatrix} \Theta\begin{pmatrix} \eta\,r \\ \xi\,t \end{pmatrix} dr\,d\eta.$$

We now readily establish the

* See Volterra: *Leçons sur les équations intégrales*, p. 74, where an equation is treated which is identical with (4), except that β is replaced by the variable s. The possibility of using the same method when one of the upper limits is constant was pointed out to me by Professor W. A. Hurwitz.

LEMMA. *If $\theta\binom{x\,s}{\xi\,t}$ is continuous in $I_{x\xi}J_{st}$ and $f(x,s)$ is continuous in $I_x J_s$, then the equation* (4) *has one and only one solution, namely*

$$u(x,s)=f(x,s)+\int_y^x\int_\alpha^\beta \Theta\binom{x\,s}{\xi\,t}f(\xi,t)\,dt\,d\xi. \tag{8}$$

Returning now to the equation (A'), let

$$\begin{aligned}\theta\binom{x\,s}{\xi\,t}&=-R\binom{x\,s}{\xi}\psi\binom{s}{\xi\,t},\\ f(x,s)&=R\binom{x\,s}{y}u(y,s)+\int_y^x R\binom{x\,s}{\xi}\lambda(\xi,s)\,d\xi,\end{aligned} \tag{9}$$

where y is regarded as a fixed point in I_x. Thus we obtain by the lemma, for each assigned function $u(y,s)$, a unique solution of (A') or (A), which, if we let

$$S\binom{x\,s}{y\,t}=\int_y^x\Theta\binom{x\,s}{\xi\,t}R\binom{\xi\,t}{y}\,d\xi, \tag{10}$$

may be put into the form

$$\begin{aligned}u(x,s)&=R\binom{x\,s}{y}u(y,s)+\int_\alpha^\beta S\binom{x\,s}{y\,t}u(y,t)\,dt\\ &\quad+\int_y^x\left[R\binom{x\,s}{\xi}\lambda(\xi,s)+\int_\alpha^\beta S\binom{x\,s}{\xi\,t}\lambda(\xi,t)\,dt\right]d\xi.\end{aligned} \tag{11}$$

Hence, we have

THEOREM I. *The integro-differential equation (A) possesses one and only one solution which reduces to the assigned initial function $u(y,s)$ at the fixed point y in I_x; this solution is given by the formula* (11).

COROLLARY I. *If the integro-differential equation (A) is homogeneous, i. e., if $\lambda(x,s)\equiv 0$, the solution has the form*

$$u(x,s)=R\binom{x\,s}{y}u(y,s)+\int_\alpha^\beta S\binom{x\,s}{y\,t}u(y,t)\,dt. \tag{12}$$

COROLLARY II. *The function*

$$w\binom{x\,s}{y}=\int_y^x\left[R\binom{x\,s}{\xi}\lambda(\xi,s)+\int_\alpha^\beta S\binom{x\,s}{\xi\,t}\lambda(\xi,t)\,dt\right]d\xi \tag{13}$$

is a particular solution of the non-homogeneous equation (A), corresponding to the initial function $u(y\,s)\equiv 0$.

Observe that the integrand of the expression $w\binom{x\,s}{y}$, when regarded as a function of x and s, is a solution of the homogeneous equation (A) for each constant value of ξ. Thus the particular solution $w\binom{x\,s}{y}$ of the non-homo-

geneous equation (A) is built up from the solutions of the homogeneous equation by an integration. It is clear that every other solution of the nonhomogeneous equation is obtainable by adding to the particular solution w a solution of the homogeneous equation.

We shall also have occasion to apply the following:

COROLLARY III. *The function* $\Theta\binom{x\,s}{y\,t}$, *when regarded as a function in* x *and* s, *is a solution of the homogeneous equation* (A), *corresponding to the initial function* $\Theta\binom{x\,s}{y\,t} = -\psi\binom{s}{y\,t}$ *at* y.

On account of the resolvent formula (6), we have

$$\Theta\begin{pmatrix} y\,s \\ y\,t \end{pmatrix} = \theta\begin{pmatrix} y\,s \\ y\,t \end{pmatrix}; \tag{14}$$

and, on account of the first formula (9) and formula (2),

$$\theta\begin{pmatrix} y\,s \\ y\,t \end{pmatrix} = -\psi\begin{pmatrix} s \\ y\,t \end{pmatrix}. \tag{15}$$

Consequently, by combining (9), (14), (15),

$$\theta\begin{pmatrix} x\,s \\ y\,t \end{pmatrix} = R\begin{pmatrix} x\,s \\ y \end{pmatrix}\Theta\begin{pmatrix} y\,s \\ y\,t \end{pmatrix};$$

and because of (10) the equation (6) becomes

$$\Theta\begin{pmatrix} x\,s \\ y\,t \end{pmatrix} = R\begin{pmatrix} x\,s \\ y \end{pmatrix}\Theta\begin{pmatrix} y\,s \\ y\,t \end{pmatrix} + \int_\alpha^\beta S\begin{pmatrix} x\,s \\ y\,\sigma \end{pmatrix}\Theta\begin{pmatrix} y\,\sigma \\ y\,t \end{pmatrix} d\sigma,$$

which is a solution by Corollary I.

3. The Boundary Problem

Let us now take a linear integral boundary expression of the following type:

$$\begin{aligned} U[u] \equiv{}& \alpha(s)u(a,s) + \beta(s)u(b,s) \\ &+ \int_\alpha^\beta [A(s,r)u(a,r) + B(s,r)u(b,r)]dr, \end{aligned} \tag{1}$$

where $\alpha(s)$, $\beta(s)$ are continuous functions in J_s, $A(s,r)$, $B(s,r)$ are continuous functions in J_{st}, and a, b are the end points of the interval I_x. Let us write from now on

$$L[u] \equiv \frac{\partial u(x,s)}{\partial x} + \phi(x,s)u(x,s) + \int_\alpha^\beta \psi\begin{pmatrix} s \\ x\,t \end{pmatrix} u(x,t)\,dt. \tag{2}$$

We shall consider the integro-differential boundary problems

$$(A) \quad L[u] = \lambda(x, s), \qquad (B) \quad U[u] = \gamma(s)$$

and

$$(A_0) \quad L[u] = 0, \qquad (B_0) \quad U[u] = 0.$$

It has been seen that all solutions of the non-homogeneous and the homogeneous equations (A), (A_0) are of the forms

$$(3) \quad u(x, s) = w\begin{pmatrix} x\,s \\ y \end{pmatrix} + R\begin{pmatrix} x\,s \\ y \end{pmatrix} u(y, s) + \int_\alpha^\beta S\begin{pmatrix} x\,s \\ y\,t \end{pmatrix} u(y, t)\,dt,$$

$$(4) \quad u(x, s) = R\begin{pmatrix} x\,s \\ y \end{pmatrix} u(y, s) + \int_\alpha^\beta S\begin{pmatrix} x\,s \\ y\,t \end{pmatrix} u(y, t)\,dt,$$

respectively, where y is a fixed point in the interval I_x at which the initial function $u(y, s)$ is to be assigned. Both y and the continuous function $u(y, s)$ are arbitrary. But in order to satisfy the boundary condition, it is clear that the initial function must be suitably chosen.

Substituting in (1) the value of $u(x, s)$ from (3), we find that the boundary condition (B) reduces to

$$(5) \quad g(y, s)u(y, s) + \int_\alpha^\beta G\begin{pmatrix} s \\ y\,t \end{pmatrix} u(y, t)\,dt = \gamma(s) - U\left[w\begin{pmatrix} x\,s \\ y \end{pmatrix}\right],$$

where

$$(6) \quad \begin{aligned} g(y, s) &= \alpha(s)R\begin{pmatrix} a\,s \\ y \end{pmatrix} + \beta(s)R\begin{pmatrix} b\,s \\ y \end{pmatrix}, \\ G\begin{pmatrix} s \\ y\,t \end{pmatrix} &= A(s, t)R\begin{pmatrix} a\,t \\ y \end{pmatrix} + B(s, t)R\begin{pmatrix} b\,t \\ y \end{pmatrix} + U\left[S\begin{pmatrix} x\,s \\ y\,t \end{pmatrix}\right]. \end{aligned}$$

This is an integral equation for determining the initial function $u(y, s)$.

Likewise, the equation (B_0) of the homogeneous system reduces to the homogeneous integral equation

$$(7) \quad g(y, s)u(y, s) + \int_\alpha^\beta G\begin{pmatrix} s \\ y\,t \end{pmatrix} u(y, t)\,dt = 0.$$

Now we impose the further condition that $\alpha(s)$ and $\beta(s)$ be such that $g(y, s)$ do not vanish at any point of J_s, so that the equations (5) and (7) may be reduced to integral equations of the second kind. Let us, then, examine this condition a little further by allowing the point y in the expression $g(y, s)$ to vary in I_y. Now

$$g(y, s) = \alpha(s)e^{-\int_y^a \varphi(\xi, s)d\xi} + \beta(s)e^{-\int_y^b \varphi(\xi, s)d\xi}.$$

If $g(y, s) \neq 0$ for a particular value of y, then

$$\alpha(s) + \beta(s)e^{-\int_a^b \varphi(\xi, s)d\xi} \neq 0,$$

i. e.,

$$\alpha(s) + \beta(s) R\binom{b\,s}{a} \neq 0. \tag{C}$$

Conversely, if (C) is fulfilled, then we shall have $g(y, s) \neq 0$ throughout J_s for each value of y in I_y. Thus the condition (C) and the condition $g(y, s) \neq 0$ are equivalent conditions, but it should be noticed that condition (C) does not involve y. Hereafter we shall always assume that (C) is fulfilled.

Under the condition (C) the equations (5) and (7) become

$$u(y, s) = F(y, s) + \int_\alpha^\beta K\binom{s}{y\,t} u(y, t)\,dt, \tag{5'}$$

$$u(y, s) = \int_\alpha^\beta K\binom{s}{y\,t} u(y, t)\,dt, \tag{7'}$$

where

$$\begin{aligned} K\binom{s}{y\,t} &= -\frac{G\binom{s}{y\,t}}{g(y, s)}, \\ F(y, s) &= \frac{\gamma(s) - U\left[w\binom{x\,s}{y}\right]}{g(y, s)}. \end{aligned} \tag{8}$$

The problem of solving the system (A, B) or (A_0, B_0) then reduces to the determination of the initial function $u(y, s)$ from the equation (5') or (7'). The initial function so determined will give the solution of the system upon substituting into the equation (3) or (4).

As in the theory of differential equations, the homogeneous system (A_0, B_0) is said to be *incompatible* if it possesses no non-trivial solution; it is said to have *compatibility of the kth order* or *index k* if there are precisely k linearly independent solutions.

Suppose $u_1(x, s), \cdots, u_n(x, s)$ are linearly dependent solutions of the homogeneous system (A_0, B_0). Then there exist constants $c_1, \cdots, c_n$, not all zero, such that

$$c_1 u_1(x, s) + \cdots + c_n u_n(x, s) = 0$$

identically in $I_x J_s$; in particular,

$$c_1 u_1(y, s) + \cdots + c_n u_n(y, s) = 0$$

for a particular value y in I_x. Conversely, if the initial functions $u_i(y, s)$ are linearly dependent, the solutions of the system formed by means of (4) will be linearly dependent. Hence

THEOREM I. *Under the condition (C), a necessary and sufficient condition*

that the homogeneous system (A_0, B_0) *have index* k *is that the integral equation* (7′) *have index** k.

THEOREM I′. *A necessary and sufficient condition that the system* (A_0, B_0), *subject to the condition* (C), *be incompatible is that the Fredholm determinant* $D(y)$ *of the kernel* $K\left({}_{y}{}^{s}_{t}\right)$ *of the equation* (7′) *does not vanish*†; *if* $D(y)$ *vanishes, the index of the system is finite.*

Since the condition (C) does not depend on y, the kernel $K\left({}_{y}{}^{s}_{t}\right)$ of (5′) and (7′) and the Fredholm determinant $D(y)$ exist for every y in I_y. Consequently, *if* $D(y_0) \neq 0$, the homogeneous system (A_0, B_0) will be incompatible and therefore the equation (7′) can have only the trivial solution $u(y, s) \equiv 0$ for every y. Hence

THEOREM II. *The Fredholm determinant,* $D(y)$, *of the equations* (5′) *and* (7′) *either vanishes everywhere in* I_y *or else vanishes nowhere.*

Every solution of (A_0, B_0) becomes, when x is changed into y, a solution of (7′). Conversely, however, a solution of (7′) will not, in general, when y is changed to x, become a solution of (A_0, B_0), as is seen by the fact that a solution of (7′) may be multiplied by an arbitrary function of y while a solution of (A_0, B_0) cannot be multiplied by an arbitrary function of x. Consequently, (A_0, B_0) is not, in general, equivalent to (7′). It is, however, equivalent in the special case in which (A_0, B_0) is incompatible, since then (7′) also has no solution except zero. In this case (A, B) has one and only one solution, which must, therefore, be the unique solution of (5′). Hence

THEOREM III. *The systems* (A, B), (A_0, B_0) *are respectively equivalent to the integral equations* (5′), (7′) *whenever the homogeneous system* (A_0, B_0) *is incompatible; when* (A_0, B_0) *is compatible, they are equivalent to* (5′), (7′) *together with the auxiliary equations* (3), (4) *respectively.*

COROLLARY. *When the homogeneous system* (A_0, B_0) *is incompatible, the non-homogeneous system* (A, B) *has a unique solution, which is given by*

$$u(x, s) = F(x, s) + \int_\alpha^\beta Q\begin{pmatrix} & s \\ x & t \end{pmatrix} F(x, t)\,dt, \tag{9}$$

where $Q\left({}_{x}{}^{s}_{t}\right)$ *is the resolvent function of the kernel* $K\left({}_{x}{}^{s}_{t}\right)$ *of the equations* (5′), (7′). *When the homogeneous system is compatible, the non-homogeneous system* (A, B) *possesses solutions if and only if*

$$\int_\alpha^\beta \phi_i(x, s) F(x, s)\,ds = 0 \tag{10}$$

for all solutions $\phi_i(x, s)$ *of the equation*

$$\phi(x, s) = \int_\alpha^\beta \phi(x, t) K\begin{pmatrix} & t \\ x & s \end{pmatrix} dt. \tag{11}$$

* Bôcher: *An Introduction to the Study of Integral Equations*, p. 45.

† Here the quantity y is regarded as a constant, but arbitrary.

4. Integro-Linear Independence

A linear integral expression

$$U[u;s] \equiv \alpha(s)u_1(s) + \beta(s)u_2(s) + \int_\alpha^\beta [A(s,r)u_1(r) + B(s,r)u_2(r)]\,dr \tag{1}$$

is said to be *integro-linearly self-dependent* in the interval J_s (or simply *self-dependent*) if there exists a continuous function $c(s)$ in J_s, not identically zero, such that

$$\int_\alpha^\beta c(s)U[u;s]\,ds = 0 \tag{2}$$

for every pair of continuous functions $u_1(s)$, $u_2(s)$; otherwise, it is said to be *self-independent.* Two linear integral expressions U_1, U_2 of the type (1) are said to be *integro-linearly dependent* (or simply *dependent*) if there exist continuous functions $c_1(s)$, $c_2(s)$, not both identically zero, such that

$$\int_\alpha^\beta (c_2(s)U_1[u;s] + c_1(s)U_2[u;s])\,ds = 0 \tag{3}$$

for every pair of continuous functions $u_1(s)$ and $u_2(s)$; otherwise they are said to be *independent.*

The above definitions of independence and self-independence are obviously a generalization of the notion of linear independence for a system of algebraic expressions. We shall derive some necessary and sufficient conditions for such dependence.

Theorem I. *A necessary and sufficient condition that the expression* (1) *be self-dependent is that the equations*

$$\begin{aligned} \alpha(s)c(s) + \int_\alpha^\beta c(r)A(r,s)\,dr &= 0, \\ \beta(s)c(s) + \int_\alpha^\beta c(r)B(r,s)\,dr &= 0 \end{aligned} \tag{4}$$

have a non-trivial solution $c(s)$ in common. This function $c(s)$ then satisfies (2), *and conversely every function $c(s)$ which satisfies* (2) *also satisfies* (4).

The theorem is an immediate result of (2) when we observe that $u_1(s)$ and $u_2(s)$ are arbitrary functions.

Corollary. *A sufficient condition that $U[u;s]$ be self-independent is that either one of the equations* (4) *possess no non-trivial solution.*

Theorem II. *When the U of formula* (1), § 3, *is self-dependent, the homogeneous system (A_0, B_0) of* § 3, *subject to the condition (C) throughout J_s as considered, has always a non-trivial solution.*

By hypothesis there exists a continuous function $c(s)$, not identically zero in J_s, which satisfies (2) and forms a common solution of the equations (4). Multiplying (4) by $R\binom{a\,s}{y}$ and $R\binom{b\,s}{y}$ respectively and adding the results together, we have, by (6), § 3,

$$g(y,s)c(s)+\int_\alpha^\beta c(r)\left\{G\binom{r}{y\,s}-U\left[S\binom{x\,r}{y\,s}\right]\right\}dr=0, \tag{5}$$

and this, by (8), § 3 and by (2) reduces to

$$g(y,s)c(s)=\int_\alpha^\beta c(r)g(y,r)K\binom{r}{y\,s}dr.$$

As a solution of (7′), § 3, we have, then,

$$u(y,s)=g(y,s)c(s).$$

Consequently the system (A_0, B_0) has a non-trivial solution.

Theorem III. *If the homogeneous system* (A_0, B_0), *subject to the condition* (C), *is compatible and if the expression* U *is self-independent, not every semi-homogeneous system*

$$(A, B_0) \qquad L[u]=\lambda(x,s), \qquad U[u]=0$$

possesses a solution.

Suppose the system (A, B_0) does possess a solution for every $\lambda(x,s)$. Then, by the Corollary to Theorem III, § 3,

$$\int_\alpha^\beta \phi_i(y,s)F(y,s)\,ds=0$$

for every solution of the equation (11), § 3. This equation reduces to

$$\int_\alpha^\beta \frac{\phi_i(y,s)}{g(y,s)}U\left[w\binom{x\,s}{y}\right]ds=0$$

because of (8), § 3. Expanding U and substituting for $w\binom{x\,s}{y}$ its value from (13), § 2, we find

$$\int_y^a\int_\alpha^\beta\left[R\binom{a\,s}{\xi}\Phi_1(y,s)+\int_\alpha^\beta S\binom{a\,s}{\xi\,t}\Phi_1(y,t)\,dt\right]\lambda(\xi,s)\,ds\,d\xi$$
$$+\int_y^b\int_\alpha^\beta\left[R\binom{b\,s}{\xi}\Phi_2(y,s)+\int_\alpha^\beta S\binom{b\,s}{\xi\,t}\Phi_2(y,t)\,dt\right]\lambda(\xi,s)\,ds\,d\xi=0,$$

where

$$\Phi_1(y,s)=\alpha(s)\frac{\phi_i(y,s)}{g(y,s)}+\int_\alpha^\beta\frac{\phi_i(y,r)}{g(y,r)}A(r,s)\,dr,$$

$$\Phi_2(y,s)=\beta(s)\frac{\phi_i(y,s)}{g(y,s)}+\int_\alpha^\beta\frac{\phi_i(y,r)}{g(y,r)}B(r,s)\,dr.$$

We now find the following equations by assuming *first* that $\lambda(\xi, s)$ is zero when $\xi \geqq y$ while it is still arbitrary when $\xi < y$, and *secondly* that it is zero when $\xi \leqq y$ and arbitrary when $\xi > y$:

$$R\begin{pmatrix} a\, s \\ \xi \end{pmatrix}\Phi_1(y, s) + \int_\alpha^\beta S\begin{pmatrix} a\, t \\ \xi\, s \end{pmatrix}\Phi_1(y, t)\,dt = 0 \quad (a \leqq \xi \leqq y),$$

$$R\begin{pmatrix} b\, s \\ \xi \end{pmatrix}\Phi_2(y, s) + \int_\alpha^\beta S\begin{pmatrix} b\, t \\ \xi\, s \end{pmatrix}\Phi_2(y, t)\,dt = 0 \quad (y \leqq \xi \leqq b).$$

Letting $\xi = a$ in the first equation and $\xi = b$ in the second, we obtain

$$\Phi_1(y, s) = 0, \qquad \Phi_2(y, s) = 0.$$

Consequently $[\phi_i(y, s)]/[g(y, s)]$ is a solution of (4) which does not vanish identically since ϕ_i may be assumed not to be identically zero. Therefore, by Theorem I, U is self-dependent, which is contrary to hypothesis.

THEOREM IV. *A necessary condition that two linear expressions U_1, U_2 of the type* (1) *be independent is that each expression be self-independent.*

THEOREM V. *A necessary and sufficient condition that the self-independent expressions U_1, U_2 be dependent on one another is that the equations*

$$\begin{aligned}(6)\qquad &\alpha_1(s)c_2(s) + \alpha_2(s)c_1(s) + \int_\alpha^\beta [c_2(r)A_1(r, s) \\ &\qquad\qquad + c_1(r)A_2(r, s)]\,dr = 0, \\ &\beta_1(s)c_2(s) + \beta_2(s)c_1(s) + \int_\alpha^\beta [c_2(r)B_1(r, s) \\ &\qquad\qquad + c_1(r)B_2(r, s)]\,dr = 0\end{aligned}$$

possess a common non-trivial solution $c_1(s)$, $c_2(s)$. These functions $c_1(s)$, $c_2(s)$ then satisfy (3); *and conversely every pair of functions $c_1(s)$, $c_2(s)$ which satisfy* (3) *also satisfy* (6).

These theorems follow immediately from the definitions of dependence and independence so that no proof will be needed.

Let us now consider, more in detail, the case in which, for every value of s in J_s,

$$(D)\qquad \Delta(s) \equiv \begin{vmatrix} \alpha_1(s) & \alpha_2(s) \\ \beta_1(s) & \beta_2(s) \end{vmatrix} \neq 0.$$

Equations (6) may now be reduced to

$$(7)\qquad c_i(s) = \int_\alpha^\beta [K_{i1}(s, t)c_1(t) + K_{i2}(s, t)c_2(t)]\,dt \quad (i = 1, 2),$$

where

$$(8)\qquad K_{ij}(s, t) = \frac{(-1)^i}{\Delta(s)}\begin{vmatrix} \alpha_i(s) & A_{3-j}(t, s) \\ \beta_i(s) & B_{3-j}(t, s) \end{vmatrix}.$$

We shall call the set of functions

$$(9)\qquad \begin{array}{ll} K_{11}(s,t), & K_{12}(s,t), \\ K_{21}(s,t), & K_{22}(s,t) \end{array}$$

the kernel-system of (7).

By the side of (7) we consider the associated non-homogeneous system

$$(10)\quad c_i(s) = f_i(s) + \int_\alpha^\beta [K_{i1}(s,t)c_1(t) + K_{i2}(s,t)c_2(t)]\,dt \quad (i=1,2).$$

We define with Fredholm two new intervals

$$J_s^{(1)}:\ \alpha_1 \leqq s \leqq \beta_1; \qquad J_s^{(2)}:\ \alpha_2 \leqq s \leqq \beta_2,$$

such that $\alpha_1 < \beta_1 \leqq \alpha_2 < \beta_2$,

$$\beta_1 - \alpha_1 = \beta_2 - \alpha_2 = \beta - \alpha.$$

Let J'_s denote the combined interval of $J_s^{(1)}$ and $J_s^{(2)}$, so that

$$\int_{J'} F(s)\,ds = \int_{J^{(1)}} F(s)\,ds + \int_{J^{(2)}} F(s)\,ds.$$

We have also four square regions $J_{st}^{(i,j)}$ $(i,j=1,2)$ to consider, and J'_{st} will be used to indicate the totality of all the squares. Then we will map our functions into the new intervals and regions in such a way that the functions K_{ij} composing the kernel system each occupy one of the four squares. That is, we define

$$(11)\qquad \begin{array}{lll} K(s,t) = K_{ij}(s-\alpha_i+\alpha,\ t-\alpha_j+\alpha) & \text{for} & J_{st}^{ij}, \\ f(s) = f_i(s-\alpha_i+\alpha) & \text{for} & J_s^i, \\ c(s) = c_i(s-\alpha_i+\alpha) & \text{for} & J_s^i \quad (i,j=1,2). \end{array}$$

According to this notation, the equations (11) have the form

$$(12)\qquad c(s) = f(s) + \int_{J'} K(s,t)c(t)\,dt.$$

This equation may be treated as an ordinary Fredholm equation by forming the Fredholm determinant, Δ, and first minor, $\Delta(s,t)$, of the kernel $K(s,t)$ in the usual way. We shall call Δ the determinant of the kernel-system (9).

If $\Delta \neq 0$, the resolvent function

$$Q(s,t) = \frac{\Delta(s,t)}{\Delta},$$

satisfies the relations

$$(13)\qquad \begin{aligned} Q(s,t) &= K(s,t) + \int_{J'} Q(s,\sigma)K(\sigma,t)\,d\sigma, \\ Q(s,t) &= K(s,t) + \int_{J'} K(s,\sigma)Q(\sigma,t)\,d\sigma; \end{aligned}$$

and the equation (12) has one and only one solution,

$$(14)\qquad c(s) = f(s) + \int_{J'} Q(s,t)f(t)\,dt.$$

Returning to the old coördinates, we can define

$$Q_{ij}(s,t) = Q(s-\alpha+\alpha_i,\ t-\alpha+\alpha_j) \qquad (\text{for } J^{ij}_{st}).$$

This is called the resolvent-system of the kernel-system K_{ij}. Equations (13) become

$$(15)\qquad \begin{aligned} Q_{ij}(s,t) = K_{ij}(s,t) + \int_\alpha^\beta [\,&Q_{i1}(s,\sigma)K_{1j}(\sigma,t) \\ &+ Q_{i2}(s,\sigma)K_{2j}(\sigma,t)\,]\,d\sigma, \\ Q_{ij}(s,t) = K_{ij}(s,t) + \int_\alpha^\beta [\,&K_{i1}(s,\sigma)Q_{1j}(\sigma,t) \\ &+ K_{i2}(s,\sigma)Q_{2j}(\sigma,t)\,]\,d\sigma, \end{aligned}$$

and the solution (14) takes the form

$$(16)\qquad c_i(s) = f_i(s) + \int_\alpha^\beta [\,Q_{i1}(s,t)f_1(t) + Q_{i2}(s,t)f_2(t)\,]\,dt \quad (i=1,2).$$

We now easily infer the truth of the following lemmas:

Lemma I. *A necessary and sufficient condition that the system* (10) *possess a unique solution is that* $\Delta \neq 0$. *If this condition is satisfied, the solution is given by formula* (16); *and, in particular, the trivial solution will be the only solution of the homogeneous system* (7).

Lemma II. *When* $\Delta = 0$, *the system* (7) *always possesses non-trivial solutions; and a necessary and sufficient condition that the system* (10) *have solutions is that the equation*

$$(17)\qquad \int_{J'} \psi(s)f(s)\,ds = \int_\alpha^\beta [\,\psi_1(s)f_1(s) + \psi_2(s)f_2(s)\,]\,ds = 0$$

be satisfied by every solution $\psi(s)$ *of the equation*

$$(18)\qquad \psi(s) = \int_{J'} \psi(t)K(t,s)\,dt.$$

By combining Lemma I and Theorem V we have

THEOREM VI. *If U_1, U_2 are self-independent and fulfill the condition (D) for every value of S in J_s, then a necessary and sufficient condition that they be independent of each other is that the Fredholm determinant Δ of the kernel-system* (9) *be different from zero.*

THEOREM VII. *If U_1, U_2 fulfill the condition (D), the equations*

$$(19) \qquad U_1 = \phi_1(s), \qquad U_2 = \phi_2(s),$$

when regarded as equations in $u_1(s)$ and $u_2(s)$, possess a unique solution if and only if U_1, U_2 are independent. This solution is integro-linear in ϕ_1 and ϕ_2.

For, if we replace u_1 and u_2 by

$$v_1(s) = \alpha_2(s)u_1(s) + \beta_2(s)u_2(s),$$

$$v_2(s) = \alpha_1(s)u_1(s) + \beta_1(s)u_2(s),$$

and let $f_1 = \phi_2$, $f_2 = \phi_1$, equations (19) become

$$v_i(s) = f_i(s) + \int_\alpha^\beta [v_1(r)K_{1i}(r,s) + v_2(r)K_{2i}(r,s)]\,dr \quad (i = 1, 2),$$

or simply

$$v(s) = f(s) + \int_{J'} v(r)K(r,s)\,dr.$$

This equation however has precisely the transposed kernel $K(r, s)$, so that it has a unique solution when and only when $\Delta \neq 0$. The second part of the theorem now follows readily.

COROLLARY I. *If (D) is fulfilled, the homogeneous equations $U_1 = 0$, $U_2 = 0$ possess non-trivial solutions when and only when U_1, U_2 are dependent.*

COROLLARY II. *If U_1 is such that $\alpha_1(s)$, $\beta_1(s)$ do not vanish together* and if $U_1 = 0$ admits no non-trivial solution, then every self-independent U_2 which fulfills (D) is independent of U_1.*

An important application of this corollary is that for a given self-independent U_1 in which $\alpha_1(s)$, $\beta_1(s)$ do not vanish together, if there can be found a U_2 such that (D) is satisfied and for which $U_2 = 0$ admits no solution other than the trivial one, then U_1, U_2 are independent. Unfortunately, I have as yet been unable to determine whether such a U_2 always exists. We shall have to leave this important general problem without giving a definite answer. Instead we shall only show the following fact which includes several important special cases in which we know U_2 can be found.

* Obviously (D) cannot be fulfilled if $\alpha_1(s)$, $\beta_1(s)$ do vanish together. On the other hand, if $\alpha_1(s)$, $\beta_1(s)$ do not vanish together, there always exist functions $\alpha_2(s)$, $\beta_2(s)$ such that (D) is fulfilled, for instance $\alpha_2 = -\beta_1$, $\beta_2 = \alpha_1$.

THEOREM VIII. *If for a given self-independent expression*

$$U_1[u] \equiv \alpha_1(s)u_1(s) + \beta_1(s)u_2(s) + \int_\alpha^\beta [A_1(s,r)u_1(r) + B_1(s,r)u_2(r)]\,dr,$$

in which $\alpha_1(s)$, $\beta_1(s)$ do not vanish together, there can be found constants k_1, k_2 such that $k_1\alpha_1(s)$, $k_2\beta_1(s)$ do not vanish together and such that

$$U_1'[u] \equiv k_1\alpha_1(s)u_1(s) + k_2\beta_1(s)u_2(s) + \int_\alpha^\beta [k_1 A_1(s,r)u_1(r) + k_2 B_1(s,r)u_2(r)]\,dr = 0$$

admits no non-trivial solution, then it is possible to find a U_2 such that (D) is fulfilled and that U_1, U_2 are independent.

For, suppose k_1, k_2 are both different from zero, then the theorem is obvious, because if we group the constants k_1, k_2 with the unknown functions $u_1(s)$, $u_2(s)$ respectively, then U_1' will have exactly the same form as U_1 so that they both can have no solution. By Corollary II, a U_2 exists.

If $k_2 = 0$, then we must have $k_1 \neq 0$, $\alpha_1(s) \neq 0$ for every value of s in J_s, and, further, on dividing U_1' by k_1, the equation

$$U_1 \equiv \alpha_1(s)u_1(s) + \int_\alpha^\beta A_1(s,r)u_1(r)\,dr = 0$$

has no non-trivial solution $u_1(s)$. Now if we define

$$U_2 \equiv \alpha_1(s)u_2(s) + \int_\alpha^\beta A_1(s,r)u_2(r)\,dr,$$

then $U_2 = 0$ will have no non-trivial solution $u_2(s)$, so that the equations $U_1 = 0$, $U_2 = 0$ together will admit only the trivial solution $u_1(s) = 0$, $u_2(s) = 0$. The same argument will enable us to construct a U_2 for the case $k_1 = 0$.

5. The Adjoint Integro-Differential Expression

DEFINITION. *The integro-differential expressions*

$$L[u] \equiv \frac{\partial u(x,s)}{\partial x} + \phi(x,s)u(x,s) + \int_\alpha^\beta \psi\begin{pmatrix} s \\ x\ t \end{pmatrix} u(x,t)\,dt,$$

$$M[v] \equiv -\frac{\partial v(x,s)}{\partial x} + \phi(x,s)v(x,s) + \int_\alpha^\beta \psi\begin{pmatrix} t \\ x\ s \end{pmatrix} v(x,t)\,dt$$

are said to be adjoint to each other; the equations

$$(A_0) \qquad L[u] = 0,$$

$$(\bar{A}_0) \qquad M[v] = 0$$

are called adjoint equations.

If we multiply $L[u]$, $M[v]$ respectively by $v(x, s)$ and $u(x, s)$, integrate with respect to s, and subtract the results, we find

$$(1) \quad \int_\alpha^\beta \big[v(x, s) L[u] - u(x, s) M[v] \big] ds = \frac{\partial}{\partial x} \int_\alpha^\beta u(x, s) v(x, s) ds,$$

which may be called Lagrange's Identity. Integrating again, with respect to x, we have the Green's theorem:

$$(2) \quad \begin{aligned} &\int_{x_1}^{x_2} \int_\alpha^\beta \big[vL[u] - uM[v] \big] ds dx \\ &\qquad = \int_\alpha^\beta [u(x_2, s) v(x_2, s) - u(x_1, s) v(x_1, s)] ds. \end{aligned}$$

These relations hold for any continuous functions $u(x, s)$ and $v(x s)$, provided they have continuous first derivatives with respect to x.

Let us write for convenience $(\bar{A}_0)$ in the form

$$(\bar{A}_0') \quad - M[v] \equiv \frac{\partial v(x, s)}{\partial x} + \bar{\phi}(x, s) v(x, s) + \int_\alpha^\beta \bar{\psi}\begin{pmatrix} & s \\ x & t \end{pmatrix} v(x, t) dt = 0.$$

A dash above a function will be used here consistently to indicate the corresponding function of the adjoint equation. The solution of $(\bar{A}_0')$ may then be written

$$(3) \qquad v(x, s) = \bar{R}\begin{pmatrix} x\, s \\ y \end{pmatrix} v(y, s) + \int_\alpha^\beta \bar{S}\begin{pmatrix} x\, s \\ y\, t \end{pmatrix} v(y, t) dt.$$

There are important symmetrical relations between the functions

$$R\begin{pmatrix} x\, s \\ y \end{pmatrix}, S\begin{pmatrix} x\, s \\ y\, t \end{pmatrix}, \bar{R}\begin{pmatrix} x\, s \\ y \end{pmatrix}, \text{ and } \bar{S}\begin{pmatrix} x\, s \\ y\, t \end{pmatrix}.$$

To obtain such relations, let us apply Green's theorem to the solutions of (A_0) and $(\bar{A}_0')$. For such functions, u, v, Green's theorem becomes

$$(4) \qquad \int_\alpha^\beta u(x_1, s) v(x_1, s) ds = \int_\alpha^\beta u(x_2, s) v(x_2, s) ds$$

for any pair of values x_1, x_2 in I_x.

Let x_3, x_4 be respectively the points at which the initial functions of u, v

are to be assigned. Then, by (12), § 2, the solutions have the forms

$$u(x,s) = R\begin{pmatrix} x & s \\ x_3 & \end{pmatrix} u(x_3,s) + \int_\alpha^\beta S\begin{pmatrix} x & s \\ x_3 & t \end{pmatrix} u(x_3,t)\,dt,$$

$$v(x,s) = \bar{R}\begin{pmatrix} x & s \\ x_4 & \end{pmatrix} v(x_4,s) + \int_\alpha^\beta \bar{S}\begin{pmatrix} x & s \\ x_4 & t \end{pmatrix} v(x_4,t)\,dt.$$

Substituting into (4) and regrouping the terms, we find

$$\int_\alpha^\beta f(s)\,u(x_3,s)\,ds = 0,$$

where

$$\begin{aligned} f(s) = {} & \left[R\begin{pmatrix} x_1 & s \\ x_3 & \end{pmatrix}\bar{R}\begin{pmatrix} x_1 & s \\ x_4 & \end{pmatrix} - R\begin{pmatrix} x_2 & s \\ x_3 & \end{pmatrix}\bar{R}\begin{pmatrix} x_2 & s \\ x_4 & \end{pmatrix}\right] v(x_4,s) \\ & + \int_\alpha^\beta \left[R\begin{pmatrix} x_1 & s \\ x_3 & \end{pmatrix}\bar{S}\begin{pmatrix} x_1 & s \\ x_4 & t \end{pmatrix} + \bar{R}\begin{pmatrix} x_1 & t \\ x_4 & \end{pmatrix} S\begin{pmatrix} x_1 & t \\ x_3 & s \end{pmatrix}\right. \\ & + \int_\alpha^\beta S\begin{pmatrix} x_1 & r \\ x_3 & s \end{pmatrix}\bar{S}\begin{pmatrix} x_1 & r \\ x_4 & t \end{pmatrix} dr - R\begin{pmatrix} x_2 & s \\ x_3 & \end{pmatrix}\bar{S}\begin{pmatrix} x_2 & s \\ x_4 & t \end{pmatrix} \\ & \left. - \bar{R}\begin{pmatrix} x_2 & t \\ x_4 & \end{pmatrix} S\begin{pmatrix} x_2 & t \\ x_3 & s \end{pmatrix} - \int_\alpha^\beta S\begin{pmatrix} x_2 & r \\ x_3 & s \end{pmatrix}\bar{S}\begin{pmatrix} x_2 & r \\ x_4 & t \end{pmatrix} dr\right] v(x_4,t)\,dt. \end{aligned}$$

Since the initial function $u(x_3,s)$ is arbitrary, we conclude that $f(s) \equiv 0$. Moreover, the initial function $v(x_4,s)$ is also arbitrary, so that, by the lemma to be proved presently, we obtain the following identities:

$$\begin{aligned} & R\begin{pmatrix} x_1 & s \\ x_3 & \end{pmatrix}\bar{R}\begin{pmatrix} x_1 & s \\ x_4 & \end{pmatrix} = R\begin{pmatrix} x_2 & s \\ x_3 & \end{pmatrix}\bar{R}\begin{pmatrix} x_2 & s \\ x_4 & \end{pmatrix}, \\ & R\begin{pmatrix} x_1 & s \\ x_3 & \end{pmatrix}\bar{S}\begin{pmatrix} x_1 & s \\ x_4 & t \end{pmatrix} + \bar{R}\begin{pmatrix} x_1 & t \\ x_4 & \end{pmatrix} S\begin{pmatrix} x_1 & t \\ x_3 & s \end{pmatrix} + \int_\alpha^\beta S\begin{pmatrix} x_1 & r \\ x_3 & s \end{pmatrix}\bar{S}\begin{pmatrix} x_1 & r \\ x_4 & t \end{pmatrix} dr \\ & \quad = R\begin{pmatrix} x_2 & s \\ x_3 & \end{pmatrix}\bar{S}\begin{pmatrix} x_2 & s \\ x_4 & t \end{pmatrix} + \bar{R}\begin{pmatrix} x_2 & t \\ x_4 & \end{pmatrix} S\begin{pmatrix} x_2 & t \\ x_3 & s \end{pmatrix} \\ & \qquad + \int_\alpha^\beta S\begin{pmatrix} x_2 & r \\ x_3 & s \end{pmatrix}\bar{S}\begin{pmatrix} x_2 & r \\ x_4 & t \end{pmatrix} dr. \end{aligned} \tag{5}$$

These relations hold identically in $I_{x_1x_2x_3x_4}J_s$ and $I_{x_1x_2x_3x_4}J_{st}$ respectively. In particular, if we let $x = x_1 = x_3$, $y = x_2 = x_4$, we have*

$$\bar{R}\begin{pmatrix} x & s \\ y & \end{pmatrix} = R\begin{pmatrix} y & s \\ x & \end{pmatrix}, \qquad \bar{S}\begin{pmatrix} x & s \\ y & t \end{pmatrix} = S\begin{pmatrix} y & t \\ x & s \end{pmatrix}. \tag{6}$$

* The first relation (6), and also the first relation (7), may also be inferred from the definition of R; see (2), § 2.

Letting $x = x_1$, $y = x_2 = x_3 = x_4$, we have

$$(7)\qquad \begin{gathered} R\begin{pmatrix} x\,s \\ y \end{pmatrix}\bar{R}\begin{pmatrix} x\,s \\ y \end{pmatrix} = 1, \\ R\begin{pmatrix} x\,s \\ y \end{pmatrix}\bar{S}\begin{pmatrix} x\,s \\ y\,t \end{pmatrix}+\bar{R}\begin{pmatrix} x\,t \\ y \end{pmatrix}S\begin{pmatrix} x\,t \\ y\,s \end{pmatrix}+\int_\alpha^\beta S\begin{pmatrix} x\,r \\ y\,s \end{pmatrix}\bar{S}\begin{pmatrix} x\,r \\ y\,t \end{pmatrix}dr=0. \end{gathered}$$

A special case of interest is when $L[u]$ is anti-self-adjoint, i. e., when $L[u] \equiv -M[u]$. In this case, we must have

$$(8)\qquad \phi(x, s) \equiv 0, \qquad \psi\begin{pmatrix} s \\ x\,t \end{pmatrix} \equiv -\psi\begin{pmatrix} t \\ x\,s \end{pmatrix}.$$

Consequently, we have

$$(9)\qquad R\begin{pmatrix} x\,s \\ y \end{pmatrix} \equiv 1, \qquad S\begin{pmatrix} x\,s \\ y\,t \end{pmatrix} \equiv \bar{S}\begin{pmatrix} x\,s \\ y\,t \end{pmatrix} = S\begin{pmatrix} y\,t \\ x\,s \end{pmatrix}.$$

We will now prove the lemma which we have referred to, and which will be useful again later.

Lemma. *If $h(s)$ and $H(s, t)$ are continuous functions such that*

$$(10)\qquad h(s)\phi(s) + \int_\alpha^\beta H(s, t)\phi(t)\,dt \equiv 0$$

for every continuous function $\phi(s)$, then $h(s) \equiv 0$ and $H(s, t) \equiv 0$.

It is sufficient to show $h(s_0) = 0$ when $\alpha < s_0 < \beta$, because it will then follow from the continuity of h that $h(s) \equiv 0$, and therefore $H(s\,t) \equiv 0$.

Let s_0 be any interior point of the interval J_s. Let a particular function $\phi(s)$ be defined as follows:

$$\phi(s) = \begin{cases} 0 & \text{for} \quad |s - s_0| > \epsilon, \\ 1 & \text{for} \quad s = s_0, \\ \text{continuous, positive and} \leqq 1 & \text{for} \quad |s - s_0| \leqq \epsilon. \end{cases}$$

Then from (10)

$$h(s_0) + \int_{s_0-\epsilon}^{s_0+\epsilon} H(s_0, t)\phi(t)\,dt = 0.$$

By the first law of the mean,

$$h(s_0) + 2\epsilon H(s_0, t_1)\phi(t_1) = 0,$$

where $s_0 - \epsilon < t_1 < s_0 + \epsilon$. Let $|H(s, t)| < M$, then

$$|h(s_0)| = 2\epsilon\phi(t_1)\,|H(s_0, t_1)| < 2\epsilon M;$$

hence $h(s_0) = 0$.

6. A Modified Form for Green's Theorem*

Let $U_1[u;s]$, $U_2[u;s]$ be the two integro-linear forms

$$(1)\quad \begin{aligned} U_i[u;s] \equiv{} & \alpha_i(s)u(a,s) + \beta_i(s)u(b,s) \\ & + \int_\alpha^\beta [A_i(s,r)u(a,r) + B_i(s,r)u(b,r)]dr \quad (i=1,2). \end{aligned}$$

Regarding (1) as equations in $u(a,s)$ and $u(b,s)$, and $U_1[u;s]$, $U_2[u;s]$ as known functions of s, it is seen (Theorem VII, § 4) that if the condition (D) is fulfilled, it is possible to solve for $u(a,s)$, $u(b,s)$ uniquely in terms of U_1 and U_2, provided the forms are independent; and furthermore, that the unique solution will consist of two integro-linear forms in U_1 and U_2 of the same form as (1). In this case, the second member of Green's Theorem ((2), § 5), in which we put $x_1 = a$ and $x_2 = b$, thus becomes

$$(2)\quad \begin{aligned} & \int_\alpha^\beta [u(b,s)v(b,s) - u(a,s)v(a,s)]ds \\ & \qquad = \int_\alpha^\beta (U_1[u;s]V_2[v;s] + U_2[u;s]V_1[v;s])ds, \end{aligned}$$

where $V_1[v;s]$, $V_2[v;s]$ are integro-linear forms in $v(a,s)$ and $v(b,s)$ of the form

$$(3)\quad \begin{aligned} V_i[v;s] \equiv{} & \gamma_i(s)v(a,s) + \delta_i(s)v(b,s) \\ & + \int_\alpha^\beta [C_i(s,r)v(a,r) + D_i(s,r)v(b,r)]dr \quad (i=1,2). \end{aligned}$$

Thus we see that Green's theorem may always be written in the form

$$(4)\quad \begin{aligned} & \int_a^b \int_\alpha^\beta \big(v(x,s)L[u] - u(x,s)M[v]dsdx\big) \\ & \qquad = \int_\alpha^\beta \big(U_1[u;s]V_2[v;s] + U_2[u;s]V_1[v;s]\big)ds \end{aligned}$$

if U_1 and U_2 are independent and satisfy condition (D).

Now, suppose (D) is not satisfied, or that (D) is satisfied but U_1 and U_2 are dependent. Will it be still possible to determine V_1 and V_2 so that the identity (2) will hold? Let us find the conditions under which V_1, V_2 can be determined so as to satisfy (2).

Assuming that U_1, U_2 have the form (1) and V_1, V_2 the form (3), let us then determine the continuous functions γ_i, δ_i, C_i, D_i $(i = 1, 2)$ so that

* In connection with §§ 6, 7 see the corresponding developments for differential equations given in these Transactions by Birkhoff, vol. 9 (1908), p. 373, and Bôcher, vol. 14 (1913), p. 415.

(2) holds for every set of continuous functions $u(a, s)$, $u(b, s)$, $v(a, s)$, $v(b, s)$. It may be remarked here that the notations $U_i[u; s]$, $U_i[u]$, $U_i(s)$, U_i will be used indiscriminately for convenience, the same being true for V_i.

On substituting (1) in (2) and equating the coefficients of the arbitrary functions $u(a, s)$ and $u(b, s)$, we obtain

$$
(5)\quad
\begin{aligned}
&\alpha_2(s) V_1[v; s] + \alpha_1(s) V_2[v; s] + \int_\alpha^\beta \big(V_1[v; r] A_2(r, s) \\
&\qquad\qquad + V_2[v; r] A_1(r, s) \big) dr = -v(a, s), \\
&\beta_2(s) V_1[v; s] + \beta_1(s) V_2[v; s] + \int_\alpha^\beta \big(V_1[v; r] B_2(r, s) \\
&\qquad\qquad + V_2[v; r] B_1(r, s) \big) dr = v(b, s)
\end{aligned}
$$

as a necessary and sufficient condition that U_1, U_2 defined by (1) should satisfy (2). Substituting in these equations the expressions for V_1 and V_2 from (3) and collecting the coefficients of the arbitrary functions $v(a, s)$ and $v(b, s)$, we find, by the lemma proved at the end of § 5, that the following identities give a necessary and sufficient condition for V_1, V_2 as defined by (3) to satisfy (5):

$$
(6a)\quad
\begin{aligned}
\alpha_2(s)\gamma_1(s) + \alpha_1(s)\gamma_2(s) + 1 &= 0, \\
\beta_2(s)\gamma_1(s) + \beta_1(s)\gamma_2(s) &= 0,
\end{aligned}
$$

$$
(6b)\quad
\begin{aligned}
\alpha_2(s)\delta_1(s) + \alpha_1(s)\delta_2(s) &= 0, \\
\beta_2(s)\delta_1(s) + \beta_1(s)\delta_2(s) - 1 &= 0,
\end{aligned}
$$

$$
(7a)\quad
\begin{aligned}
&\alpha_2(s) C_1(s, r) + \alpha_1(s) C_2(s, r) + \gamma_1(r) A_2(r, s) \\
&\qquad + \gamma_2(r) A_1(r, s) \\
&\qquad + \int_\alpha^\beta \big(A_2(t, s) C_1(t, r) + A_1(t, s) C_2(t, r) \big) dt = 0, \\
&\beta_2(s) C_1(s, r) + \beta_1(s) C_2(s, r) + \gamma_1(r) B_2(r, s) \\
&\qquad + \gamma_2(r) B_1(r, s) \\
&\qquad + \int_\alpha^\beta \big(B_2(t, s) C_1(t, r) + B_1(t, s) C_2(t, r) \big) dt = 0,
\end{aligned}
$$

$$
(7b)\quad
\begin{aligned}
&\alpha_2(s)D_1(s,r)+\alpha_1(s)D_2(s,r)+\delta_1(r)A_2(r,s)\\
&\qquad+\delta_2(r)A_1(r,s)\\
&\qquad+\int_a^\beta\big(A_2(t,s)D_1(t,r)+A_1(t,s)D_2(t,r)\big)\,dt=0,\\
&\beta_2(s)D_1(s,r)+\beta_1(s)D_2(s,r)+\delta_1(r)B_2(r,s)\\
&\qquad+\delta_2(r)B_1(r,s)\\
&\qquad+\int_a^\beta\big(B_2(t,s)D_1(t,r)+B_1(t,s)D_2(t,r)\big)\,dt=0.
\end{aligned}
$$

Thus these eight equations form a necessary and sufficient condition that U_1, U_2, V_1, V_2 *as defined by* (1) *and* (3) *satisfy* (2). We will now inquire under what conditions the continuous functions γ_i, δ_i, C_i, D_i $(i=1,2)$ can be determined to satisfy equations (6*a*), (6*b*), (7*a*), (7*b*).

If $\Delta(s)\neq 0$ for every value of s in J_s, there will be a unique solution of equations (6*a*) and (6*b*), namely

$$
(8)\qquad \gamma_i(s)=(-1)^{i-1}\frac{\beta_i(s)}{\Delta(s)},\qquad \delta_i(s)=(-1)^{i-1}\frac{\alpha_i(s)}{\Delta(s)}\quad (i=1,2).
$$

On the other hand, if for a particular value, s_0, we have $\Delta(s_0)=0$, then, in order that the matrix and the augmented matrix of the system (6*a*) have the same rank, we must have $\beta_1(s_0)=\beta_2(s_0)=0$. But this cannot be the case, as we see from the second equation (6*b*). Consequently, γ_i, δ_i cannot be determined when the condition (D) is not fulfilled. The condition (D) is then a first necessary condition that we have to impose on U_1, U_2 in order that the problem in question be possible.

Assuming then that (D) is satisfied by U_1 and U_2, let us now consider the system (7*a*).

Using the notation (8), § 4 and letting

$$
(9)\qquad
\begin{aligned}
f_i(s,r)&=\begin{vmatrix}\gamma_1(r) & K_{i2}(s,r)\\ -\gamma_2(r) & K_{i1}(s,r)\end{vmatrix},\\
g_i(s,r)&=\begin{vmatrix}\delta_1(r) & K_{i2}(s,r)\\ -\delta_2(r) & K_{i1}(s,r)\end{vmatrix},
\end{aligned}
$$

the equations (7*a*) may readily be reduced to the form

$$
(10)\qquad
\begin{aligned}
C_i(s,r)=f_i(s,r)+\int_a^\beta[&K_{i1}(s,t)C_1(t,r)\\
&+K_{i2}(s,t)C_2(t,r)]\,dt\quad (i=1,2).
\end{aligned}
$$

In this form we have precisely a system of equations of the type (10), § 4.

Now if the Fredholm determinant Δ of the kernel system $K_{ij}(s, t)$, $(i, j = 1, 2)$, is different from zero, we have by Lemma I, § 4, a unique solution of the equations, which is given by

$$C_i(s, r) = f_i(s, r) + \int_a^\beta [Q_{i1}(s, t) f_1(t, r) + Q_{i2}(s, t) f_2(t, r)]\, dt \qquad (i = 1, 2).$$

Because of (9) and the resolvent relations (15), § 4, this solution simplifies into

$$C_i(s, r) = \begin{vmatrix} \gamma_1(r) & Q_{i2}(s, r) \\ -\gamma_2(r) & Q_{i1}(s, r) \end{vmatrix} \qquad (i = 1, 2),$$

and because of (8) it further reduces to

$$C_i(s, r) = \frac{1}{\Delta(r)} \begin{vmatrix} \beta_1(r) & Q_{i2}(s, r) \\ \beta_2(r) & Q_{i1}(s, r) \end{vmatrix} \qquad (i = 1, 2). \tag{11}$$

Similarly, for the system (7*b*) we have the unique solution

$$D_i(s, r) = \frac{1}{\Delta(r)} \begin{vmatrix} \alpha_1(r) & Q_{i2}(s, r) \\ \alpha_2(r) & Q_{i1}(s, r) \end{vmatrix} \qquad (i = 1, 2). \tag{12}$$

On the other hand, if $\Delta = 0$, solutions of (7*a*), (7*b*) both exist by Lemma II, § 4, if and only if

$$\int_a^\beta [\psi_1(s) f_1(s, r) + \psi_2(s) f_2(s, r)]\, ds = 0,$$

$$\int_a^\beta [\psi_1(s) g_1(s, r) + \psi_2(s) g_2(s, r)]\, ds = 0$$

for every non-trivial solution, $\psi_1(s)$, $\psi_2(s)$, of the equations

$$\psi_i(s) = \int_a^\beta [\psi_1(t) K_{1i}(t, s) + \psi_2(t) K_{2i}(t, s)]\, dt \quad (i = 1, 2). \tag{13}$$

Suppose both of these conditions are satisfied. Substituting the values of f_1, f_2, g_1, g_2 from (9) and $\gamma_1, \gamma_2, \delta_1, \delta_2$ from (8), we have

$$\frac{\beta_1(r)}{\Delta(r)} \int_a^\beta [\psi_1(s) K_{11}(s, r) + \psi_2(s) K_{21}(s, r)]\, ds - \frac{\beta_2(r)}{\Delta(r)} \int_a^\beta [\psi_1(s) K_{12}(s, r) + \psi_2(s) K_{22}(s, r)]\, ds = 0,$$

$$\frac{\alpha_1(r)}{\Delta(r)} \int_a^\beta [\psi_1(s) K_{11}(s, r) + \psi_2(s) K_{21}(s, r)]\, ds - \frac{\alpha_2(r)}{\Delta(r)} \int_a^\beta [\psi_1(s) K_{12}(s, r) + \psi_2(s) K_{22}(s, r)]\, ds = 0.$$

These equations may now be regarded as a system of linear algebraic equations whose determinant, $\Delta(r)$, does not vanish for any value of r in J_r. Whence

$$\int_a^\beta [\psi_1(s)K_{1i}(s, r) + \psi_2(s)K_{2i}(s, r)]\,ds = 0 \qquad (i = 1, 2),$$

i. e., $\psi_1 \equiv 0$, $\psi_2 \equiv 0$ because of (13). But this is contrary to the fact that $\psi_1(s)$, $\psi_2(s)$ are a non-trivial solution of (13). Hence C_i, D_i cannot be determined when $\Delta = 0$. Thus we have $\Delta \neq 0$ as a second necessary condition to be imposed on U_1, U_2; that is (§ 4, Theorem VI), U_1, U_2 must be independent in addition to fulfilling the condition (D). Hence

THEOREM I. *A necessary and sufficient condition that the expressions V_1, V_2 of the type* (3) *be determinable so that the identity* (2) *holds for every set of continuous functions $u(a, s)$, $u(b, s)$, $v(a, s)$, $v(b, s)$, is that U_1, U_2 fulfill condition (D) and that they be independent. The determination is unique and given by formulas* (3), (8), (11), (12).

Now let us suppose that we start from the expressions V_1, V_2 just determined and that we try to determine U_1, U_2 so as to satisfy (3). We form the determinants $\bar{\Delta}(s)$, $\bar{\Delta}$ for the expressions V_1, V_2 corresponding to the determinants $\Delta(s)$, Δ for U_1, U_2, and denote by $(\bar{D})$ the condition that, for every value of s in J_s, $\bar{\Delta}(s) \neq 0$. Then, by the theorem just stated, since U_1, U_2 do exist, we have the

COROLLARY I. *If U_1, U_2 are independent and fulfill the condition (D), then the expressions V_1, V_2 are also independent and fulfill the condition $(\bar{D})$. Thus in this case the two sets of expressions are uniquely determinable from each other.*

We see that the necessary and sufficient condition of Theorem I is precisely a necessary and sufficient condition that the system $U_1 = 0$, $U_2 = 0$, admit no non-trivial solution (Theorem VII, Corollary I, § 4). Hence

COROLLARY II. *If U_1, U_2 are such that the system $U_1 = 0$, $U_2 = 0$, admits no non-trivial solution, then V_1, V_2 can be determined and they are such that the system $V_1 = 0$, $V_2 = 0$, admits no non-trivial solution.*

The following fact will be useful later.

COROLLARY III. *If V_1, V_2 exist, then $u(a, s)$, $u(b, s)$ can be uniquely expressed in terms of U_1 and U_2 in the form*

$$\begin{aligned} u(a, s) &= -\gamma_2(s)U_1(s) - \gamma_1(s)U_2(s) \\ &\qquad - \int_a^\beta [U_1(r)C_2(r, s) + U_2(r)C_1(r, s)]\,dr, \\ u(b, s) &= \delta_2(s)U_1(s) + \delta_1(s)U_2(s) \\ &\qquad + \int_a^\beta [U_1(r)D_2(r, s) + U_2(r)D_1(r, s)]\,dr. \end{aligned} \tag{14}$$

The existence of a unique solution follows from Theorem VII, § 4, and there it is also shown that the solution is integro-linear in U_1 and U_2. Thus we need now only to verify the formulæ (14). For this purpose we assume

$$u(a, s) = \alpha_1'(s) U_1(s) + \alpha_2'(s) U_2(s) + \int_\alpha^\beta [A_1'(s, r) U_1(r) + A_2'(s, r) U_2(r)] dr,$$

$$u(b, s) = \beta_1'(s) U_1(s) + \beta_2'(s) U_2(s) + \int_\alpha^\beta [B_1'(s, r) U_1(r) + B_2'(s, r) U_2(r)] dr.$$

Substituting in the expression

$$\int_\alpha^\beta [u(b, s) v(b, s) - u(a, s) v(a, s)] ds$$

and collecting the coefficients of U_1 and U_2, we find

$$\int_\alpha^\beta [u(b, s) v(b, s) - u(a, s) v(a, s)] ds = \int_\alpha^\beta [U_1(s) V_2'(s) + U_2(s) V_1'(s)] ds, \tag{15}$$

where

$$V_i'(s) = -\alpha_i'(s) v(a, s) + \beta_i'(s) v(b, s) - \int_\alpha^\beta [v(a, r) A_i'(r, s) - v(b, r) B_i'(r, s)] dr. \tag{16}$$

But (15) is exactly the identity (2). Since we have seen that for each given set of U_1 and U_2, the expressions V_1, V_2 are uniquely determined and are given by (3), the expressions (16) and (13) are identical. Hence we have the formulas (14).

7. The Adjoint System

It has been seen that the expressions V_1, V_2 are uniquely determined for each U_2 integro-linearly independent of U_1 and fulfilling (D). Now let U_2' be another expression independent of U_1 and let V_1', V_2' be the corresponding expressions thereby determined. We are to see how the two sets of V_i are related to one another.

The two sets of expressions, U_1, U_2, V_1, V_2, and U_1, U_2', V_1', V_2', satisfy the identity (2) of § 6. Consequently,

$$\int_\alpha^\beta \big((U_1[u; s] V_2[v; s] + U_2[u; s] V_1[v; s] \big) ds = \int_\alpha^\beta \big(U_1[u; s] V_2'[v; s] + U_2'[u; s] V_1'[v; s] \big) ds.$$

Let U_1, U_2, U_2' be written in their full form and the coefficients of the arbitrary functions $u(a, s)$, $u(b, s)$ be equated to zero; we obtain

$$
(1)\quad
\begin{aligned}
&\alpha_1(s)V_2[v;s] + \alpha_2(s)V_1[v;s] \\
&\qquad + \int_\alpha^\beta \big(A_1(r,s)V_2[v;r] + A_2(r,s)V_1[v;r]\big)\,dr \\
&\quad = \alpha_1(s)V_2'[v;s] + \alpha_2'(s)V_1'[v;s] \\
&\qquad + \int_\alpha^\beta \big(A_1(r,s)V_2'[v;r] + A_2'(r,s)V_1'[v;r]\big)\,dr, \\
&\beta_1(s)V_2[v;s] + \beta_2(s)V_1[v;s] \\
&\qquad + \int_\alpha^\beta \big(B_1(r,s)V_2[v;r] + B_2(r,s)V_1[v;r]\big)\,dr \\
&\quad = \beta_1(s)V_2'[v;s] + \beta_2'(s)V_1'[v;s] \\
&\qquad + \int_\alpha^\beta \big(B_1(r,s)V_2'[v;r] + B_2'(r,s)V_1'[v;r]\big)\,dr.
\end{aligned}
$$

Let us denote by $\Phi[V';s]$, $\Psi[V';s]$ respectively the expressions on the right of these equations. Since $\Delta(s) \neq 0$, we find

$$
V_i[v;s] = F_i[V';s] + \int_\alpha^\beta \big(K_{i1}(s,r)V_1[v;r] + K_{i2}(s,r)V_2[v;r]\big)\,dr \qquad (i = 1, 2),
$$

where

$$
F_i[V';s] = \frac{(-1)^{i-1}}{\Delta(s)} \begin{vmatrix} \alpha_i(s) & \Phi[V';s] \\ \beta_i(s) & \Psi[V';s] \end{vmatrix}.
$$

The expressions F_1, F_2 are integro-linear and homogeneous in V_1' and V_2'. As $\Delta \neq 0$, these equations may be solved for V_1 and V_2, and the solution is unique, having the form

$$
V_i[v;s] = F_i[V';s] + \int_\alpha^\beta \big(Q_{i1}(s,r)F_1[V';r] + Q_{i2}(s,r)F_2[V';r]\big)\,dr \qquad (i = 1, 2).
$$

The expressions in the second member are obviously integro-linear and homogeneous in V_1' and V_2', so that these equations may be regarded as an integro-linear transformation between the expressions V_1, V_2 and V_1', V_2'. Upon simplifications due to the resolvent relations (15) of § 4, these equations take the final form

$$
(2)\quad
\begin{aligned}
V_1[v;s] &= M_1(s)V_1'[v;s] + \int_\alpha^\beta N_1(s,r)V_1'[v;r]\,dr, \\
V_2[v;s] &= V_2'[v;s] + M_2(s)V_1'[v;s] + \int_\alpha^\beta N_2(s,r)V_1'[v;r]\,dr.
\end{aligned}
$$

Similar equations may be obtained for expressing V_1', V_2' in terms of V_1, V_2 by solving (1) for V_1', V_2'. It is important to notice that both of these integro-linear transformations are unique, since all the coefficients depend only on the coefficients of U_1, U_2, and U_2'.

The importance of the equations (2) lies in the fact that V_1 is integro-linear and homogeneous in V_1', so that whenever the boundary condition $V_1' = 0$ is satisfied, the condition $V_1 = 0$ is also satisfied, and vice versa. For this reason, we may state:

THEOREM I. *The condition $V_1 = 0$ is essentially determined by the condition $U_1 = 0$, and conversely.*

DEFINITION. A pair of boundary conditions $U_1 = 0$, $V_1 = 0$ are said to be *adjoint* to each other if U_1, V_1 satisfy a relation of the form (2), § 6, where U_2 is independent of U_1 and the condition $\Delta(s) \neq 0$ is fulfilled. The systems

$$(A_0) \quad L[u] = 0, \qquad (B_0) \quad U_1[u] = 0,$$

$$(\bar{A}_0) \quad M[v] = 0, \qquad (\bar{B}_0) \quad V_1[v] = 0$$

are called adjoint systems.

It follows from Theorem VIII, § 4, that an adjoint boundary condition always exists if the function $\alpha_1(s)$ does not vanish in J_s and the Fredholm determinant of $[-A_1(s, r)]/[\alpha_1(s)]$ is not zero; and also under certain more general conditions there specified.

As we have done in § 3, we will restrict ourselves to the case in which the system (A_0, B_0) is subject to the condition

$$(C) \qquad \alpha_1(s) + R\begin{pmatrix} b & s \\ a & \end{pmatrix}\beta_1(s) \neq 0.$$

If we consider the adjoint system $(\bar{A}_0, \bar{B}_0)$, we find that a similar condition

$$(\bar{C}) \qquad \gamma_1(s) + \bar{R}\begin{pmatrix} b & s \\ a & \end{pmatrix}\delta_1(s) \neq 0$$

is fulfilled. For, from the formulæ (8), § 6 and (6), (7), § 5, we have

$$\gamma_1(s) + \bar{R}\begin{pmatrix} b & s \\ a & \end{pmatrix}\delta_1(s) = \frac{\beta_1(s) + R\begin{pmatrix} a & s \\ b & \end{pmatrix}\alpha_1(s)}{\Delta(s)}$$

$$= \frac{\alpha_1(s) + R\begin{pmatrix} b & s \\ a & \end{pmatrix}\beta_1(s)}{\Delta(s)R\begin{pmatrix} b & s \\ a & \end{pmatrix}}.$$

Hence

THEOREM II. *If the system (A_0, B_0) fulfills the condition (C), then the adjoint system $(\bar{A}_0, \bar{B}_0)$ fulfills a similar condition $(\bar{C})$.*

We shall next prove

THEOREM III. *The adjoint systems* (A_0, B_0), $(\bar{A}_0, \bar{B}_0)$, *subject to the condition* (C), *have the same index.*

Let n be the index of the system (A_0, B_0) and m that of the adjoint system $(\bar{A}_0, \bar{B}_0)$. Let $u_1, \cdots, u_n$ and $v_1, \cdots, v_m$ be respectively complete sets of linearly independent solutions of the systems. Let u be any solution of the equation (A_0), and v any solution of $(\bar{A}_0)$.

Applying Green's theorem to u and v_i, we have

$$\int_\alpha^\beta U_1[u; s] V_2[v_i; s]\, ds = 0 \qquad (i = 1, 2, \cdots, m) \tag{3}$$

for all solutions u of (A_0). As before, let y be any fixed value of x at which the initial function $u(y, s)$ is assigned. Then, by formula (7), § 3,

$$U_1[u; s] = g(y, s) u(y, s) + \int_\alpha^\beta G\begin{pmatrix} & s \\ y & r \end{pmatrix} u(y, r)\, dr,$$

where $g(y, s)$ and $G\left(\begin{smallmatrix} & s \\ y & r\end{smallmatrix}\right)$ are given by (6), § 3. Because (3) has to hold for all continuous functions $u(y, s)$, we have

$$g(y, s) V_2[v_i; s] + \int_\alpha^\beta V_2[v_i; r] G\begin{pmatrix} & r \\ y & s \end{pmatrix} dr = 0,$$

or

$$g(y, s) V_2[v_i; s] = \int_\alpha^\beta g(y, r) V_2[v_i; r] K\begin{pmatrix} & r \\ y & s \end{pmatrix} dr.$$

That is to say, $\phi_i(y, s) = g(y, s) V_2[v_i; s]$, $(i = 1, 2, \cdots, m)$, are solutions of the equation

$$\phi(y, s) = \int_\alpha^\beta \phi(y, r) K\begin{pmatrix} & r \\ y & s \end{pmatrix} dr. \tag{4}$$

On the other hand, the initial functions of $u_1, \cdots, u_n$ form a complete set of linearly independent solutions of the equation

$$u(y, s) = \int_\alpha^\beta K\begin{pmatrix} & s \\ y & r \end{pmatrix} u(y, r)\, dr \tag{5}$$

adjoint to (4). Hence if it can be shown (as we will now do) that the functions $V_2[v_i; s]$ $(i = 1, 2, \cdots, m)$ are linearly independent, then $\phi_i(y, s)$ will constitute m linearly independent solutions of (4), and therefore $m \leqq n$.

For, suppose $V_2[v_i; s]$ $(i = 1, 2, \cdots, m)$ were linearly dependent. Then there would exist constants $c_1, \cdots, c_m$, not all zero, such that

$$c_1 V_2[v_1; s] + \cdots + c_m V_2[v_m; s] = 0.$$

Let us define

$$v_0 = c_1 v_1 + \cdots + c_m v_m.$$

Then v_0 is also a non-trivial solution of the system $(\bar{A}_0, \bar{B}_0)$, and satisfies

$$V_2[v_0; s] = c_1 V_2[v_1; s] + \cdots + c_m V_2[v_m; s] = 0.$$

But this is contradictory, because $V_1 = 0$, $V_2 = 0$ admit no non-trivial solution. This completes the proof that $m \leqq n$.

In the same manner, the functions $\psi_i(y, s) = \bar{g}(y, s) U_2[u_i; s]$ $(i = 1, 2, \cdots, n)$ form n linearly independent solutions of the equation

$$\psi(y, s) = \int_a^\beta \psi(y, r) \bar{K}\begin{pmatrix} r \\ y\, s \end{pmatrix} dr, \tag{6}$$

where $\bar{K}$ has the same meaning in the adjoint system $(\bar{A}_0, \bar{B}_0)$ as K has in the original system (A_0, B_0). On the other hand, the initial functions of $v_1, \cdots, v_m$ form a complete set of linearly independent solutions of the equation

$$v(y, s) = \int_a^\beta \bar{K}\begin{pmatrix} s \\ x\, y \end{pmatrix} v(y, r)\, dr \tag{7}$$

adjoint to (6). Hence $n \leqq m$. When combined with the previous result, we have $m = n$. Thus we have established Theorem III and also

THEOREM IV. *Let $u_1, \cdots, u_n$ be a complete set of linearly independent solutions of the system (A_0, B_0), and $v_1, \cdots, v_n$ a complete set of linearly independent solutions of the adjoint system $(\bar{A}_0, \bar{B}_0)$. Then the functions*

$$\phi_i(y, s) = g(y, s) V_2[v_i; s] \qquad (i = 1, 2, \cdots, n) \tag{8}$$

form a complete set of linearly independent solutions of the equation (4), *and the functions*

$$\psi_i(y, s) = \bar{g}(y, s) U_2[u_i; s] \qquad (i = 1, 2, \cdots, n) \tag{9}$$

form a complete set of linearly independent solutions of the equation (6).

If we replace $\phi_i(y, s)$ in the equation (10) of § 3 by the values (8), we obtain from the second part of the Corollary of Theorem III, § 3, the

THEOREM V. *A necessary and sufficient condition that a non-homogeneous system (A, B), subject to the condition (C), possess a solution when the reduced system (A_0, B_0) is compatible and when the adjoint system $(\bar{A}_0, \bar{B}_0)$ exists, is that*

$$\int_a^\beta F(y, s) g(y, s) V_2[v_i; s]\, ds \equiv 0 \tag{10}$$

for every v_i which satisfies the adjoint system $(\bar{A}_0, \bar{B}_0)$.

By means of (8), § 3, this condition may be given the form

$$\int_a^\beta \left(\gamma(s) - U_1\left[w\begin{pmatrix} x\, s \\ y \end{pmatrix} \right] \right) V_2[v_i; s]\, ds \equiv 0. \tag{11}$$

8. The Self-Adjoint Boundary Conditions

We have shown that two different choices of the auxiliary boundary expression U_2 independent of the given U_1 and fulfilling the condition (D) lead to two expressions V_1 which are connected by an integro-linear transformation. Furthermore, this transformation is unique in both ways. This fact is important for us here, because in seeking the conditions that a given expression U_1 be self-adjoint, it is sufficient to seek the conditions that a particular V_1 thereby determined be connected with U_1 by an integro-linear transformation.* It is clear that if one particular V_1 is integro-linearly connected with U_1, then every V_1 will be so connected.

Suppose the condition $U_1 = 0$ is self-adjoint, and that, for a particular choice of U_2, we have

$$V_1[u; s] = M(s) U_1[u; s] + \int_\alpha^\beta N(s, t) U_1[u; t] \, dt, \tag{1}$$

where

$$U_1[u; s] = \alpha_1(s) u(a, s) + \beta_1(s) u(b, s) + \int_\alpha^\beta [A_1(s, r) u(a, r) + B_1(s, r) u(b, r)] \, dr,$$

$$V_1[u; s] = \gamma_1(s) u(a, s) + \delta_1(s) u(b, s) + \int_\alpha^\beta [C_1(s, r) u(a, r) + D_1(s, r) u(b, r)] \, dr,$$

the functions γ_1, δ_1, C_1, D_1 having the values given by the equations (8), (11), and (12) of § 6. The equation (1) may be thrown into the following form

$$\begin{aligned} V_1[u; s] = {} & M(s) \alpha_1(s) u(a, s) + M(s) \beta_1(s) u(b, s) \\ & + \int_\alpha^\beta [M(s) A_1(s, r) + N(s, r) \alpha_1(r) \\ & \qquad + \int_\alpha^\beta N(s, t) A_1(t, r) \, dt] u(a, r) \, dr \\ & + \int_\alpha^\beta [M(s) B_1(s, r) + N(s, r) \beta_1(r) \\ & \qquad + \int_\alpha^\beta N(s, t) B_1(t, r) \, dt] u(b, r) \, dr, \end{aligned}$$

whereby we obtain

$$\gamma_1(s) = M(s) \alpha_1(s), \tag{2}$$

* Professor D. Jackson takes the same point of view in his article, in these Transactions, vol. 17 (1916), pp. 418–424.

$$(2') \qquad \delta_1(s) = M(s)\beta_1(s),$$

$$(3) \quad C_1(s, r) = M(s)A_1(s, r) + N(s, r)\alpha_1(r) + \int_a^\beta N(s, t)A_1(t, r)\,dt,$$

$$(3') \quad D_1(s, r) = M(s)B_1(s, r) + N(s, r)\beta_1(r) + \int_a^\beta N(s, t)B_1(t, r)\,dt.$$

Equations (2) and (3) may be regarded as determining the functions M and N; equations (2′) and (3′) then constitute the conditions which must be imposed on U_1 in order that it be self-adjoint.

Substituting the values from (8), § 6, for γ_1 and δ_1, the equations (2), (2′) become

$$\Delta(s)M(s)\alpha_1(s) - \beta_1(s) = 0,$$

$$\alpha_1(s) - \Delta(s)M(s)\beta_1(s) = 0.$$

Now $\alpha_1(s)$ and $\beta_1(s)$ cannot vanish together since we must have $\Delta(s) \neq 0$ throughout J_s in order that the adjoint expressions exist. Hence, we must have

$$\begin{vmatrix} \Delta(s)M(s) & 1 \\ 1 & \Delta(s)M(s) \end{vmatrix} = 0$$

for every value of s in J. That is,

$$M(s) = \pm\frac{1}{\Delta(s)}.$$

It follows that

$$(4) \qquad \alpha_1(s) = \pm\beta_1(s) \neq 0 \qquad \text{(throughout } J_s\text{)}.$$

Conversely, when (4) is satisfied,

$$M(s) = \pm\frac{1}{\Delta(s)}.$$

Equation (4) is a first necessary condition.

Assuming then that (4) is satisfied, let us proceed to consider the equations (3), (3′). These equations are Fredholm equations with the kernels $-[A_1(t, r)]/[\alpha_1(r)]$ and $-[B_1(t, r)]/[\beta_1(r)]$ respectively. It is conceivable that the Fredholm determinant of either one of these kernels might be zero. We shall now show that in such case, no self-adjoint system is possible.

Let us suppose the Fredholm determinant of the kernel $-[A_1(t, r)]/[\alpha_1(r)]$ to be zero, and a self-adjoint expression to exist so that the equation (3) has a solution. For this it is necessary that

$$(5) \qquad \int_a^\beta \phi(r)\left[\frac{C_1(s, r)}{\alpha_1(r)} - \frac{1}{\pm\Delta(s)}\frac{A_1(s, r)}{\alpha_1(r)}\right]dr = 0$$

for every non-trivial solution $\phi(r)$ of the transposed equation

$$\phi(r) = \int_\alpha^\beta \left[-\frac{A_1(r,t)}{\alpha_1(t)} \right] \phi(t)\,dt. \tag{6}$$

The $\pm$ signs correspond to those of (4). Because of (6), the condition (5) has the form

$$\frac{\phi(s)}{\pm\Delta(s)} + \int_\alpha^\beta \frac{C_1(s,r)}{\alpha_1(r)} \phi(r)\,dr = 0,$$

or, since

$$\gamma_1(s) = \frac{\beta_1(s)}{\Delta(s)} = \frac{\pm\alpha_1(s)}{\Delta(s)},$$

$$\gamma_1(s)\frac{\phi(s)}{\alpha_1(s)} + \int_\alpha^\beta C_1(s,r)\frac{\phi(r)}{\alpha_1(r)}\,dr = 0. \tag{7}$$

Let us put

$$u(a,s) = \frac{\phi(s)}{\alpha_1(s)}, \qquad u(b,s) = 0.$$

Then equations (6) and (7) become

$$U_1[u;s] = 0, \qquad V_1[u;s] = 0.$$

The identity (2), § 6 now becomes, if we let $u(x,s) \equiv v(x,s)$,

$$-\int_\alpha^\beta \left[\frac{\phi(s)}{\alpha_1(s)} \right]^2 ds = 0,$$

whence $\phi(s) \equiv 0$. But this is contrary to the fact that $\phi(s)$ is a non-trivial solution of (6). Thus we have derived a second necessary condition for the existence of self-adjoint expressions, namely, that both* $-[A_1(t,r)]/[\alpha_1(r)]$ and $-[B_1(t,r)]/[\beta_1(r)]$ have non-vanishing Fredholm determinants. This condition is sufficient to insure the existence of a unique solution for each of the equations (3), (3′), and we shall have a third necessary condition upon equating these solutions to each other. It is also clear that these three necessary conditions combined are also sufficient for the existence of self-adjoint expressions.

To determine the explicit form of the third condition, it is convenient to choose a particular U_2 which will simplify the computation. We shall choose for instance U_2 such that

$$\alpha_2(s) \equiv 0, \qquad \beta_2(s) \equiv 1, \qquad A_2(s,r) \equiv B_2(s,r) \equiv 0.$$

This U_2 is integro-linearly independent of the given U_1. To prove this we

* The proof for $-[B_1(t,r)]/[\beta_1(r)]$ proceeds exactly in the same way.

note that $\Delta(s) = \alpha_1(s) \neq 0$ in J_s; and according to the notations of § 4,

$$K_{11}(s, r) = K_{21}(s, r) = 0,$$

$$K_{22}(s, r) = -\frac{A_1(r, s)}{\alpha_1(s)}, \qquad \text{(throughout } J_{sr}\text{)}.$$

Consequently $K(s, r) = 0$ whenever the second argument r is in the interval $J^{(1)}$. Let us write for short

$$\mathfrak{A}(s, r) = -\frac{A_1(r, s)}{\alpha_1(s)}, \qquad \mathfrak{B}(s, r) = -\frac{B_1(r, s)}{\beta_1(s)},$$

and denote their resolvent functions, which by hypothesis exist, by $\mathfrak{A}'(s, r)$, $\mathfrak{B}'(s, r)$. Then

$$\Delta = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{J'} \overset{(n)}{\cdots} \int_{J'} K\begin{pmatrix} s_1 \cdots s_n \\ s_1 \cdots s_n \end{pmatrix} ds_1 \cdots ds_n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_a^\beta \overset{(n)}{\cdots} \int_a^\beta K_{22}\begin{pmatrix} s_1 \cdots s_n \\ s_1 \cdots s_n \end{pmatrix} ds_1 \cdots ds_n .$$

This is different from zero, because it is precisely the Fredholm determinant of the kernel $\mathfrak{A}(s, r)$. This completes the proof that U_2 is integro-linearly independent of U_1 by Theorem VI, § 4.

From the resolvent relations (15), § 4, we have also the following further facts:

$$Q_{11}(s, r) = Q_{21}(s, r) = 0,$$

$$Q_{22}(s, r) = K_{22}(s, r) + \int_a^\beta Q_{22}(s, t) K_{22}(t, r)\, dt,$$

$$Q_{22}(s, r) = K_{22}(s, r) + \int_a^\beta K_{22}(s, t) Q_{22}(t, r)\, dt,$$

$$Q_{12}(s, r) = K_{12}(s, r) + \int_a^\beta K_{12}(s, t) Q_{22}(t, r)\, dt,$$

whence $Q_{22}(s, r) = \mathfrak{A}'(s, r)$. Furthermore,

$$C_1(s, r) = -\frac{Q_{12}(s, r)}{\alpha_1(r)}, \qquad D_1(s, r) = 0.$$

If we let $\alpha_1(s) = \pm \beta_1(s) = 1$, as we may do without loss of generality, we have

$$\Delta(s) = 1, \qquad \mathfrak{A}(s, r) = -A_1(r, s), \qquad \mathfrak{B}(s, r) = \mp B_1(r, s),$$
$$K_{12}(s, r) = \pm \mathfrak{B}(s, r) \mp \mathfrak{A}(s, r),$$

$$-C_1(s, r) = Q_{12}(s, r) = \mp \mathfrak{A}'(s, r) \pm \mathfrak{B}(s, r) \pm \int_a^\beta \mathfrak{B}(s, t) \mathfrak{A}'(t, r)\, dt,$$

and the equations (3), (3′) have the following form

$$N(s,r) = C_1(s,r) \pm \mathfrak{A}(r,s) + \int_\alpha^\beta \mathfrak{A}(r,t)N(s,t)\,dt,$$

$$N(s,r) = \pm \mathfrak{B}(r,s) + \int_\alpha^\beta \mathfrak{B}(r,t)N(s,t)\,dt.$$

Solving,

$$\begin{aligned} N(s,r) &= [C_1(s,r) \pm \mathfrak{A}(r,s)] + \int_\alpha^\beta \mathfrak{A}'(r,t)[C_1(s,t) \pm \mathfrak{A}(t,s)]\,dt \\ &= \pm\left[\mathfrak{A}'(s,r) + \mathfrak{A}'(r,s) + \int_\alpha^\beta \mathfrak{A}'(s,t)\mathfrak{A}'(r,t)\,dt\right] \mp \mathfrak{B}(s,r) \\ &\quad \mp \int_\alpha^\beta \mathfrak{B}(s,\sigma)\left[\mathfrak{A}'(\sigma,r) + \mathfrak{A}'(r,\sigma) \right. \\ &\qquad\qquad \left. + \int_\alpha^\beta \mathfrak{A}'(\sigma,t)\mathfrak{A}'(r,t)\,dt\right]d\sigma, \end{aligned}$$

$$N(s,r) = \pm \mathfrak{B}(r,s) \pm \int_\alpha^\beta \mathfrak{B}'(r,t)\mathfrak{B}(t,s)\,dt = \pm \mathfrak{B}'(r,s).$$

Equating and transposing,

$$\begin{aligned} \left[\mathfrak{A}'(s,r) + \mathfrak{A}'(r,s) + \int_\alpha^\beta \mathfrak{A}'(s,t)\mathfrak{A}'(r,t)\,dt\right] &= \mathfrak{B}(s,r) + \mathfrak{B}'(r,s) \\ + \int_\alpha^\beta \mathfrak{B}(s,\sigma)\left[\mathfrak{A}'(\sigma,r) + \mathfrak{A}'(r,\sigma) + \int_\alpha^\beta \mathfrak{A}'(\sigma,t)\mathfrak{A}'(r,t)\,dt\right]&d\sigma. \end{aligned}$$

Upon solving and simplifying, we obtain the last condition in the final form

$$\begin{aligned} \mathfrak{A}'(s,r) + \mathfrak{A}'(r,s) + \int_\alpha^\beta \mathfrak{A}'(s,\sigma)\mathfrak{A}'(r,\sigma)\,d\sigma \\ = \mathfrak{B}'(s,r) + \mathfrak{B}'(r,s) + \int_\alpha^\beta \mathfrak{B}'(s,\sigma)\mathfrak{B}'(r,\sigma)\,d\sigma. \end{aligned} \tag{8}$$

THEOREM I. *Every self-adjoint integro-linear boundary condition may be reduced to the form*

$$\begin{aligned} U[u;s] \equiv u(a,s) \pm u(b,s) - \int_\alpha^\beta [u(a,r)\mathfrak{A}(r,s) \\ \pm u(b,r)\mathfrak{B}(r,s)]\,dr = 0, \end{aligned}$$

in which the Fredholm determinants of $\mathfrak{A}$ *and* $\mathfrak{B}$ *are not zero and their resolvent functions* $\mathfrak{A}'$ *and* $\mathfrak{B}'$ *satisfy the relation* (8). *Conversely, every condition of this form is self-adjoint, provided, of course, that the condition* (C)

$$1 \pm R\binom{b\,s}{a} \neq 0$$

is fulfilled.

COROLLARY. *When the integro-differential expression $L[u]$ is anti-self-adjoint and the boundary condition $U_1[u] = 0$ is self-adjoint, the latter must have the form*

$$U[u] \equiv u(a, s) + u(b, s) - \int_\alpha^\beta [u(a, r)\mathfrak{A}(r, s) + u(b, r)\mathfrak{B}(r, s)]\, dr = 0.$$

For, when $L[u]$ is anti-self-adjoint, $R\binom{x\,s}{y} \equiv 1$; so the condition (C) will not be satisfied when $\alpha(s) = -\beta(s) = 1$.

9. The Green's Functions

In the theory of linear differential equations the conception of the Green's functions enables us to write down in an explicit form the solution of a semi-homogeneous boundary problem consisting of a single linear differential equation of the nth order, or a system of n linear differential equations of the first order, and of a system of n homogeneous linear boundary equations, whenever the reduced system is incompatible.*

Following out this analogy, we are led to try to find a solution of a system

$$(A, B_0) \qquad L[u] = \lambda(x, s), \qquad U[u] = 0$$

in the form

$$(1) \quad u(x, s) = \int_a^b H\begin{pmatrix} x\,s \\ y \end{pmatrix} \lambda(y, s)\, dy + \int_a^b \int_\alpha^\beta G\begin{pmatrix} x\,s \\ y\,t \end{pmatrix} \lambda(y, t)\, dt\, dy,$$

where H and G are independent of λ. These two functions we shall call the system of Green's functions for (A, B_0). We may arrive at such functions by imposing certain conditions of discontinuity suggested by the discontinuities of Green's functions for differential equations.

Let $G\binom{x\,s}{y\,t}$ be continuous, together with its first partial derivative with respect to x, throughout the region $I_{xy}J_{st}$; let $H\binom{x\,s}{y}$ be continuous, together with its first partial derivative with respect to x, throughout each of the following regions:

$$T_1:\ \{a \leqq y \leqq x \leqq b,\ J_s\}, \qquad T_2:\ \{a \leqq x \leqq y \leqq b,\ J_s\};$$

finally let H possess a discontinuity when $x = y$ of the type

$$(2) \qquad H\begin{pmatrix} y+, s \\ y \end{pmatrix} - H\begin{pmatrix} y-, s \\ y \end{pmatrix} = 1.$$

* Birkhoff, these Transactions, vol. 9 (1908), p. 377; Bounitzky, Liouville's Journal, ser. 6, vol. 5 (1909), p. 65; Bôcher, Annals of Mathematics, ser. 2, vol. 13 (1911–12), p. 71.

We use the notation $H\left(\begin{smallmatrix} y\pm, s \\ y \end{smallmatrix}\right)$ to mean $\lim\limits_{\epsilon \doteq 0} H,\left(\begin{smallmatrix} y\pm\epsilon, s \\ y \end{smallmatrix}\right)$ $\epsilon > 0$; also

$$H\begin{pmatrix} y & s \\ y\pm & \end{pmatrix} = \lim_{\epsilon \doteq 0} H\begin{pmatrix} x & s \\ x\pm\epsilon & \end{pmatrix}.$$

Because of the continuity throughout T_1 and T_2, it is clear that

$$H\begin{pmatrix} y\pm, s \\ y \end{pmatrix} = H\begin{pmatrix} y & s \\ y\mp & \end{pmatrix}.$$

Replacing u by its value from (1), we find

$$L[u] = \left[H\begin{pmatrix} x & s \\ x- & \end{pmatrix} - H\begin{pmatrix} x & s \\ x+ & \end{pmatrix}\right]\lambda(x,s)$$

$$+\int_a^b \left[\frac{\partial H\begin{pmatrix} x\,s \\ y \end{pmatrix}}{\partial x} + \phi(x,s)H\begin{pmatrix} x\,s \\ y \end{pmatrix}\right]\lambda(y,s)\,dy$$

$$+\int_a^b\int_\alpha^\beta \left[L\left[G\begin{pmatrix} x\,s \\ y\,t \end{pmatrix}\right] + \psi\begin{pmatrix} s \\ x\,t \end{pmatrix}H\begin{pmatrix} x\,t \\ y \end{pmatrix}\right]\lambda(y,t)\,dt\,dy.$$

Hence, on account of (2), u as given by (1) satisfies (A) for every continuous function $\lambda(x,s)$ if and only if $H\left(\begin{smallmatrix} x\,s \\ y \end{smallmatrix}\right)$ and $G\left(\begin{smallmatrix} x\,s \\ y\,t \end{smallmatrix}\right)$ respectively satisfy the equations

$$\frac{\partial H\begin{pmatrix} x\,s \\ y \end{pmatrix}}{\partial x} + \phi(x,s)H\begin{pmatrix} x\,s \\ y \end{pmatrix} = 0, \tag{3}$$

$$L\left[G\begin{pmatrix} x\,s \\ y\,t \end{pmatrix}\right] = -\psi\begin{pmatrix} s \\ x\,t \end{pmatrix}H\begin{pmatrix} x\,t \\ y \end{pmatrix}. \tag{4}$$

Both of these equations have to be considered separately in the regions T_1 and T_2, although the formal work is the same.

We may now regard the functions

$$H\begin{pmatrix} y+, s \\ y \end{pmatrix},\ H\begin{pmatrix} y-, s \\ y \end{pmatrix},\ G\begin{pmatrix} y\,s \\ y\,t \end{pmatrix}$$

as the initial functions given at a fixed point y in the interval I_x. These functions will at present be assumed to be continuous in their respective variables, and to satisfy condition (2); otherwise they are arbitrary, pending further determination.

By (2), § 2, the solution of (3) is

$$H\begin{pmatrix} x\,s \\ y \end{pmatrix} = R\begin{pmatrix} x\,s \\ y \end{pmatrix}H\begin{pmatrix} y\pm, s \\ y \end{pmatrix}, \tag{5}$$

in which the $\pm$ signs correspond respectively to the regions T_1, T_2.

The equation (4) may be solved by the result of § 2 in the form

$$G\binom{x\,s}{y\,t} = R\binom{x\,s}{y}G\binom{y\,s}{y\,t} + \int_\alpha^\beta S\binom{x\,s}{y\,\sigma}G\binom{y\,\sigma}{y\,t}d\sigma$$

$$-\int_y^x\left[R\binom{x\,s}{\xi}\psi\binom{s}{\xi\,t}H\binom{\xi\,t}{y} + \int_\alpha^\beta S\binom{x\,s}{\xi\,\sigma}\psi\binom{\sigma}{\xi\,t}H\binom{\xi\,t}{y}d\sigma\right]d\xi.$$

By Corollary III, Theorem I, § 2, this simplifies into

$$G\binom{x\,s}{y\,t} = R\binom{x\,s}{y}G\binom{y\,s}{y\,t} + \int_\alpha^\beta S\binom{x\,s}{y\,\sigma}G\binom{y\,\sigma}{y\,t}d\sigma$$

$$+\int_y^x \Theta\binom{x\,s}{\xi\,t}H\binom{\xi\,t}{y}d\xi.$$

Replacing $H\binom{\xi\,t}{y}$ by its values from (5) and making use of the definition (10), § 2, this further simplifies into

$$(6)\qquad \begin{aligned} G\binom{x\,s}{y\,t} &= R\binom{x\,s}{y}G\binom{y\,s}{y\,t} \\ &+\int_\alpha^\beta S\binom{x\,s}{y\,\sigma}G\binom{y\,\sigma}{y\,t}d\sigma + H\binom{y\pm,t}{y}S\binom{x\,s}{y\,t},\end{aligned}$$

where the $\pm$ signs again correspond to the regions T_1 and T_2. It is important to observe that the function $G\binom{x\,s}{y\,t}$ thus determined is continuous throughout $I_{xy}J_{st}$, because the only possible place of discontinuity is when $x = y$, but then $S\binom{x\,s}{y\,t} = 0$ by virtue of its definition.

We are now to determine $H\binom{y\pm,s}{y}$ and $G\binom{y\,s}{y\,t}$ so that the expression (1) also satisfies the boundary equation (B_0) for all $\lambda(x,s)$. Upon substitution of (1) in $U[u]$ we have

$$U[u] = \int_a^b\left[\alpha(s)H\binom{a\,s}{y} + \beta(s)H\binom{b\,s}{y}\right]\lambda(y,s)\,dy$$

$$+\int_a^b\int_\alpha^\beta\left[A(s\,t)H\binom{a\,t}{y} + B(s\,t)H\binom{b\,t}{y}\right.$$

$$\left.+\,U\left[G\binom{x\,s}{y\,t}\right]\right]\lambda(y,t)\,dt\,dy.$$

We shall have $U[u] = 0$ for all $\lambda(x,s)$ if and only if the equations

$$(7)\qquad \alpha(s)H\binom{a\,s}{y} + \beta(s)H\binom{b\,s}{y} = 0,$$

$$(8)\qquad U\left[G\binom{x\,s}{y\,t}\right] + A(s\,t)H\binom{a\,t}{y} + B(s\,t)H\binom{b\,t}{y} = 0$$

are satisfied.

On substituting in (7) for $H\binom{a\,s}{y}$ and $H\binom{b\,s}{y}$ their values from (5), we obtain an equation in $H\binom{y+,s}{y}$ and $H\binom{y-,s}{y}$, which together with (2) enables us to find for these functions the values

$$(9)\quad H\begin{pmatrix} y+,s \\ y \end{pmatrix} = \frac{\alpha(s)R\begin{pmatrix} a\,s \\ y \end{pmatrix}}{g(y,s)}, \qquad H\begin{pmatrix} y-,s \\ y \end{pmatrix} = \frac{-\beta(s)R\begin{pmatrix} b\,s \\ y \end{pmatrix}}{g(y,s)},$$

since we confine ourselves to the case $g(y,s) \neq 0$ (§ 4). It is convenient at this stage to introduce the following abbreviations which will be useful later.

$$g_1(y,s) = \alpha(s)R\begin{pmatrix} a\,s \\ y \end{pmatrix}, \qquad g_2(y,s) = \beta(s)R\begin{pmatrix} b\,s \\ y \end{pmatrix},$$

$$(10)\quad G_1\begin{pmatrix} s \\ y\,t \end{pmatrix} = A(s,t)R\begin{pmatrix} a\,t \\ y \end{pmatrix} + \alpha(s)S\begin{pmatrix} a\,s \\ y\,t \end{pmatrix} + \int_\alpha^\beta A(s,\sigma)S\begin{pmatrix} a\,\sigma \\ y\,t \end{pmatrix}d\sigma,$$

$$G_2\begin{pmatrix} s \\ y\,t \end{pmatrix} = B(s,t)R\begin{pmatrix} b\,t \\ y \end{pmatrix} + \beta(s)S\begin{pmatrix} b\,s \\ y\,t \end{pmatrix} + \int_\alpha^\beta B(s,\sigma)S\begin{pmatrix} b\,\sigma \\ y\,t \end{pmatrix}d\sigma.$$

Thus according to the notations (6), § 3, we have*

$$(11)\quad g(y,s) = g_1(y,s) + g_2(y,s), \qquad G\begin{pmatrix} s \\ y\,t \end{pmatrix} = G_1\begin{pmatrix} s \\ y\,t \end{pmatrix} + G_2\begin{pmatrix} s \\ y\,t \end{pmatrix},$$

and the equations (9) may be written

$$(9')\qquad H\begin{pmatrix} y+,s \\ y \end{pmatrix} = \frac{g_1(y,s)}{g(y,s)}, \qquad H\begin{pmatrix} y-,s \\ y \end{pmatrix} = -\frac{g_2(y,s)}{g(y,s)}.$$

Now from the equation (6) we have

$$G\begin{pmatrix} a\,s \\ y\,t \end{pmatrix} = R\begin{pmatrix} a\,s \\ y \end{pmatrix}G\begin{pmatrix} y\,s \\ y\,t \end{pmatrix} + \int_\alpha^\beta S\begin{pmatrix} a\,s \\ y\,\sigma \end{pmatrix}G\begin{pmatrix} y\,\sigma \\ y\,t \end{pmatrix}d\sigma + H\begin{pmatrix} y-,t \\ y \end{pmatrix}S\begin{pmatrix} a\,s \\ y\,t \end{pmatrix},$$

$$G\begin{pmatrix} b\,s \\ y\,t \end{pmatrix} = R\begin{pmatrix} b\,s \\ y \end{pmatrix}G\begin{pmatrix} y\,s \\ y\,t \end{pmatrix} + \int_\alpha^\beta S\begin{pmatrix} b\,s \\ y\,\sigma \end{pmatrix}G\begin{pmatrix} y\,\sigma \\ y\,t \end{pmatrix}d\sigma + H\begin{pmatrix} y+,t \\ y \end{pmatrix}S\begin{pmatrix} b\,s \\ y\,t \end{pmatrix};$$

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* Note that $G\begin{pmatrix} s \\ y\,t \end{pmatrix}$ and $G\begin{pmatrix} x\,s \\ y\,t \end{pmatrix}$ are two entirely different functions.

whence

$$U\left[G\begin{pmatrix} x\,s \\ y\,t \end{pmatrix}\right] = g(y,s)\,G\begin{pmatrix} y\,s \\ y\,t \end{pmatrix} + \int_\alpha^\beta G\begin{pmatrix} s \\ y\,r \end{pmatrix} G\begin{pmatrix} y\,r \\ y\,t \end{pmatrix} dr$$

$$+ H\begin{pmatrix} y-,t \\ y \end{pmatrix}\left[\alpha(s)\,S\begin{pmatrix} a\,s \\ y\,t \end{pmatrix} + \int_\alpha^\beta A(s,\sigma)\,S\begin{pmatrix} a\,\sigma \\ y\,t \end{pmatrix} d\sigma\right]$$

$$+ H\begin{pmatrix} y+,t \\ y \end{pmatrix}\left[\beta(s)\,S\begin{pmatrix} b\,s \\ y\,t \end{pmatrix} + \int_\alpha^\beta B(s,\sigma)\,S\begin{pmatrix} b\,\sigma \\ y\,t \end{pmatrix} d\sigma\right].$$

We shall now substitute this value in (8) and also replace $H\left(\begin{smallmatrix} a\,t \\ y \end{smallmatrix}\right)$, $H\left(\begin{smallmatrix} b\,t \\ y \end{smallmatrix}\right)$ by their values from (5). If in the resulting equation we replace $H\left(\begin{smallmatrix} y+,s \\ y \end{smallmatrix}\right)$, $H\left(\begin{smallmatrix} y-,s \\ y \end{smallmatrix}\right)$ by their values from (9′), we find that (8) reduces to

$$g(y,s)\,G\begin{pmatrix} y\,s \\ y\,t \end{pmatrix} + \int_\alpha^\beta G\begin{pmatrix} s \\ y\,r \end{pmatrix} G\begin{pmatrix} y\,r \\ y\,t \end{pmatrix} dr$$

$$= \frac{1}{g(y,t)}\left[g_2(y,t)\,G_1\begin{pmatrix} s \\ y\,t \end{pmatrix} - g_1(y,t)\,G_2\begin{pmatrix} s \\ y\,t \end{pmatrix}\right],$$

or

$$G\begin{pmatrix} y\,s \\ y\,t \end{pmatrix} = F\begin{pmatrix} s \\ y\,t \end{pmatrix} + \int_\alpha^\beta K\begin{pmatrix} s \\ y\,r \end{pmatrix} G\begin{pmatrix} y\,r \\ y\,t \end{pmatrix} dr, \tag{12}$$

if we write for short

$$F\begin{pmatrix} s \\ y\,t \end{pmatrix} = \frac{-1}{g(y,s)\,g(y,t)} \begin{vmatrix} g_1(y,t) & G_1\begin{pmatrix} s \\ y\,t \end{pmatrix} \\ g_2(y,t) & G_2\begin{pmatrix} s \\ y\,t \end{pmatrix} \end{vmatrix}. \tag{13}$$

The kernel $K\left(\begin{smallmatrix} s \\ y\,r \end{smallmatrix}\right)$ is the same as that in the equation (7′), § 3.

Now if the homogeneous system (A_0, B_0) is incompatible, then the kernel $K\left(\begin{smallmatrix} s \\ y\,r \end{smallmatrix}\right)$ possesses a resolvent function $Q\left(\begin{smallmatrix} s \\ y\,r \end{smallmatrix}\right)$ and the equation (12) possesses a unique solution given by

$$G\begin{pmatrix} y\,s \\ y\,t \end{pmatrix} = F\begin{pmatrix} s \\ y\,t \end{pmatrix} + \int_\alpha^\beta Q\begin{pmatrix} s \\ y\,r \end{pmatrix} F\begin{pmatrix} r \\ y\,t \end{pmatrix} dr. \tag{14}$$

Definition. The functions $H\left(\begin{smallmatrix} x\,s \\ y \end{smallmatrix}\right)$, $G\left(\begin{smallmatrix} x\,s \\ y\,t \end{smallmatrix}\right)$ are said to form a system of Green's functions of the integro-differential boundary problem (A_0, B_0): $L[u] = 0$, $U[u] = 0$, where $U[u]$ is assumed to be integro-linearly self-independent and subject to the condition (C), if they are defined respectively in the regions $I_{xy}\,J_s$ and $I_{xy}\,J_{st}$ and possess the following properties:

1. $H\left(\begin{smallmatrix} x\,s \\ y \end{smallmatrix}\right)$ is continuous together with the first partial derivative with respect to x in the regions T_1 and T_2, and

$$H\begin{pmatrix} y+,s \\ y \end{pmatrix} - H\begin{pmatrix} y-,s \\ y \end{pmatrix} = 1.$$

2. $G\binom{x\,s}{y\,t}$ is continuous together with the first partial derivative with respect to x throughout the region $I_{xy}J_{st}$.

3. Throughout T_1 and T_2 the functions $H\binom{x\,s}{y}$ satisfies the equations (3) and (7).

4. The function $G\binom{x\,s}{y\,t}$ satisfies the equations (4) and (8).

Theorem I. *When Green's functions exist, the semi-homogeneous system* (A, B_0) *possesses a solution given by the formula* (1).

We have seen in the above deduction that Green's functions exist if the system (A_0, B_0) is incompatible. Because of the fact (Theorem III, § 4) that when (A_0, B_0) is compatible not every semi-homogeneous system (A, B_0) can have a solution, it follows that Green's functions do not exist for this case. Hence

Theorem II. *A necessary and sufficient condition that a system of Green's functions exist for a system* (A_0, B_0), *in which* U *is self-independent and* (C) *is fulfilled, is that the system* (A_0, B_0) *be incompatible. When this condition is satisfied, the solution given by* (1) *is the unique solution.*

The last fact follows from the Corollary to Theorem III, § 3. From the theorem just stated, it follows that the equation (12) cannot possess a solution whenever the system (A_0, B_0) is compatible. Thus we have the

Corollary. *When a system* (A_0, B_0), *in which* U *is self-independent and fulfills* (C), *is compatible, the function* (13) *cannot vanish identically; and*

$$\int_\alpha^\beta \phi_i(y, s) F\binom{s}{y\,t} ds$$

does not vanish identically for every $\phi_i(y, s)$ *which satisfies*

$$\phi(y, s) = \int_\alpha^\beta \phi(y, r) K\binom{r}{y\,s} dr.$$

Theorem III. *For a system* (A, B_0) *there cannot exist more than one set of functions* $H\binom{x\,s}{y}$, $G\binom{x\,s}{y\,t}$ *such that* (1) *is a solution of the system for every* $\lambda(x, s)$; *and if such a set exists, it consists of the Green's functions for the system.*

When the reduced system (A_0, B_0) is compatible, no such functions H and G can exist, because in that case not every semi-homogeneous system (A, B_0) can have a solution (Theorem III, § 4). When (A_0, B_0) is incompatible, Green's functions exist and (1) is the unique solution of (A, B_0). Hence if there exists another set of functions, H' and G', such that

$$u(x, s) = \int_a^b H'\binom{x\,s}{y} \lambda(y, s)\, dy + \int_a^b \int_\alpha^\beta G'\binom{x\,s}{y\,t} \lambda(y, t)\, dt\, dy$$

is also a solution of (A, B_0), this solution must be identical with (1) and

therefore the difference of this and (1) is identically zero. Since $\lambda(x, s)$ is arbitrary, we find, by using the lemma in § 5, $H' \equiv H$, $G' \equiv G$.

If in the system (A, B_0) we replace the boundary condition $U = 0$ by another boundary condition $U' = 0$, where U' is an integro-linear function of U, then (1) will be obviously also a solution of the resulting system; hence

COROLLARY. *The Green's functions of a system are invariant of the choice of boundary conditions, provided the different choices of boundary expressions are integro-linearly connected.*

Another important property is that there exists a symmetrical relation between the Green's functions of the given system and the adjoint system. From Corollary I, Theorem I, § 6, it follows that the adjoint boundary condition $V = 0$ is self-independent. By reference to Theorems II, III, § 7, we infer from Theorem II:

THEOREM IV. *If the system (A_0, B_0) possesses Green's functions, H, G, the adjoint system $(\bar{A}_0, \bar{B}_0)$ possesses Green's functions, $\bar{H}$, $\bar{G}$.*

The solution of the adjoint semi-homogeneous system

$$(\bar{A}, \bar{B}_0) \qquad - M[v] = \mu(x, s), \qquad V[v] = 0$$

is given by

$$v(x, s) = \int_a^b \bar{H}\begin{pmatrix} x\, s \\ y \end{pmatrix} \mu(y, s)\, dy + \int_a^b \int_\alpha^\beta \bar{G}\begin{pmatrix} x\, s \\ y\, t \end{pmatrix} \mu(y, t)\, dt\, dy. \tag{15}$$

Let $u(x, s)$ be the solution of the system (A, B_0) given by (1). Then, by Green's theorem,

$$\int_a^b \int_\alpha^\beta [v(x, s)\lambda(x, s) + u(x, s)\mu(x, s)]\, ds\, dx = 0.$$

On the substitution of the values of u and v from (1) and (15), we have

$$\int_a^b \int_a^b \int_\alpha^\beta \left[\bar{H}\begin{pmatrix} x\, s \\ y \end{pmatrix} + H\begin{pmatrix} y\, s \\ x \end{pmatrix}\right] \lambda(x, s)\mu(y, s)\, ds\, dy\, dx$$
$$+ \int_a^b \int_a^b \int_\alpha^\beta \int_\alpha^\beta \left[\bar{G}\begin{pmatrix} x\, s \\ y\, t \end{pmatrix} + G\begin{pmatrix} y\, t \\ x\, s \end{pmatrix}\right] \lambda(x, s)\mu(y, t)\, dt\, ds\, dy\, dx = 0,$$

which holds for every λ and μ. Hence, by the lemma in § 5,

$$\bar{H}\begin{pmatrix} x\, s \\ y \end{pmatrix} = - H\begin{pmatrix} y\, s \\ x \end{pmatrix}, \qquad \bar{G}\begin{pmatrix} x\, s \\ y\, t \end{pmatrix} = - G\begin{pmatrix} y\, t \\ x\, s \end{pmatrix}. \tag{16}$$

THEOREM V. *The Green's functions of adjoint systems satisfy* (16).

THEOREM VI. *If two systems*

$$(A, B_0) \quad \begin{matrix} L[u] = \lambda(x, s), \\ U_1[u] = 0, \end{matrix} \qquad\qquad (A', B_0') \quad \begin{matrix} L'[u] = \lambda(x, s), \\ U_1'[u] = 0 \end{matrix}$$

have the same Green's functions $H\binom{x\,s}{y}$, $G\binom{x\,s}{y\,t}$, *and if the adjoint system* $(\bar{A}_0, \bar{B}_0)$ *exists, then the expressions* L *and* L' *are identical and* U_1' *is an integro-linear function of* U_1.

Since the Green's functions are the same for both systems, the function formed from them by the formula (1) satisfies both systems, hence it satisfies the homogeneous equation

$$L[u] - L'[u] = 0,$$

that is,

$$\phi''(x, s)u(x, s) + \int_\alpha^\beta \psi''\begin{pmatrix} x\,s \\ t \end{pmatrix} u(x, t)\,dt = 0,$$

if we let

$$\phi''(x, s) = \phi(x, s) - \phi'(x, s),$$

$$\psi''\begin{pmatrix} s \\ x\,t \end{pmatrix} = \psi\begin{pmatrix} s \\ x\,t \end{pmatrix} - \psi'\begin{pmatrix} s \\ x\,t \end{pmatrix}.$$

If we substitute (1) in this equation, we find

$$\begin{aligned} \int_a^b \phi''(x, s) H\begin{pmatrix} x\,s \\ y \end{pmatrix} \lambda(y, s)\,dy & \\ + \int_a^b \int_\alpha^\beta \Bigg[\phi''(x, s) G\begin{pmatrix} x\,s \\ y\,t \end{pmatrix} &+ \psi''\begin{pmatrix} s \\ x\,t \end{pmatrix} H\begin{pmatrix} x\,t \\ y \end{pmatrix} \\ + \int_\alpha^\beta \psi''\begin{pmatrix} s \\ x\,r \end{pmatrix} G\begin{pmatrix} x\,r \\ y\,t \end{pmatrix} dr \Bigg] &\lambda(y, t)\,dt\,dy = 0. \end{aligned}$$

By the lemma of § 5, we obtain

$$\tag{17} \begin{gathered} \phi''(x, s) H\begin{pmatrix} x\,s \\ y \end{pmatrix} \equiv 0, \\ \phi''(x, s) G\begin{pmatrix} x\,s \\ y\,t \end{pmatrix} + \psi''\begin{pmatrix} s \\ x\,t \end{pmatrix} H\begin{pmatrix} x\,t \\ y \end{pmatrix} + \int_\alpha^\beta \psi''\begin{pmatrix} s \\ x\,r \end{pmatrix} G\begin{pmatrix} x\,r \\ y\,t \end{pmatrix} dr \equiv 0. \end{gathered}$$

Let us take the limit of the first of these equations as y approaches x first from above and then from below. This gives

$$\phi''(x, s) H\begin{pmatrix} x & s \\ x\pm & \end{pmatrix} = 0,$$

and by subtracting one of these equations from the other, we see from (2) that $\phi'' = 0$.

Substituting this value in the second equation (17), and replacing H and G by their values $-\bar{H}$, $-\bar{G}$, we find

$$\bar{H}\begin{pmatrix} y\,t \\ x \end{pmatrix} \psi''\begin{pmatrix} s \\ x\,t \end{pmatrix} + \int_\alpha^\beta \bar{G}\begin{pmatrix} y\,t \\ x\,r \end{pmatrix} \psi''\begin{pmatrix} s \\ x\,r \end{pmatrix} dr \equiv 0.$$

Hence

$$\int_a^b \bar{H}\begin{pmatrix} y\,t \\ x \end{pmatrix} \psi''\begin{pmatrix} s \\ x\,t \end{pmatrix} dx + \int_a^b \int_\alpha^\beta \bar{G}\begin{pmatrix} y\,t \\ x\,r \end{pmatrix} \psi''\begin{pmatrix} s \\ x\,r \end{pmatrix} dr\,dx \equiv 0.$$

Now the first member of this equation is, by (15), the solution of the system

$$- M[v] = \psi''\begin{pmatrix} s \\ y\,t \end{pmatrix}, \qquad V_1[v] = 0,$$

regarded as equations in y and t. Hence $\psi'' = 0$. This completes the proof that $L[u]$ and $L'[u]$ are identical.

Our theorem will be proved if we can show that U_1 and U_1' are integro-linearly connected. For this purpose we substitute in U_1' for $u(a, s)$ and $u(b, s)$ their values from (14), § 6. This gives

$$U_1'(s) = M_1(s)\,U_1(s) + \int_\alpha^\beta N_1(s, t)\,U_1(t)\,dt$$
$$+ M_2(s)\,U_2(s) + \int_\alpha^\beta N_2(s, t)\,U_2(t)\,dt,$$

in which

$$M_2(s) = -\alpha_1'(s)\gamma_1(s) + \beta_1'(s)\delta_1(s),$$
$$N_2(s, t) = -\alpha_1'(s)C_1(t, s) + \beta_1'(s)D_1(t, s) - A_1'(s, t)\gamma_1(t)$$
$$+ B_1'(s, t)\delta_1(t) + \int_\alpha^\beta [-A_1'(s, r)C_1(t, r)$$
$$+ B_1'(s, r)D_1(t, r)]\,dr.$$

And if we can show that $M_2(s) \equiv 0$, $N_2(s, t) \equiv 0$, we shall have established an integro-linear relation between U_1 and U_1'.

To prove $M_2(s) \equiv 0$, we make use of (7) and the corresponding formula for (A', B_0'). We have, since (A, B_0) and (A', B_0') have the same Green's functions,

$$\alpha_1(s)\,H\begin{pmatrix} a\,s \\ y \end{pmatrix} + \beta_1(s)\,H\begin{pmatrix} b\,s \\ y \end{pmatrix} = 0,$$

$$\alpha_1'(s)\,H\begin{pmatrix} a\,s \\ y \end{pmatrix} + \beta_1'(s)\,H\begin{pmatrix} b\,s \\ y \end{pmatrix} = 0.$$

Now for each constant value s_0 the functions $H\binom{a\ s_0}{y}$ and $H\binom{b\ s_0}{y}$ cannot both vanish identically, because otherwise we would have from (5) both $H\binom{y-,\ s_0}{y}$ and $H\binom{y+,\ s_0}{y}$ identically zero, which is impossible owing to the discontinuity of H. Consequently

$$-\alpha_1'(s)\beta_1(s) + \beta_1'(s)\alpha_1(s) \equiv 0.$$

Hence, from (8), § 6, we have $M_2 \equiv 0$.

To show $N_2(s,t) \equiv 0$, we have from the formulæ corresponding to (8) in the case of the systems $(\bar{A}, \bar{B}_0)$ and (A', B'_0)

$$-V_{yt}\left[\bar{G}\begin{pmatrix} y\,t \\ x\,s \end{pmatrix}\right] = C_1(t,s)\bar{H}\begin{pmatrix} a\,s \\ x \end{pmatrix} + D_1(t,s)\bar{H}\begin{pmatrix} b\,s \\ x \end{pmatrix},$$

$$-U'_{xs}\left[\bar{G}\begin{pmatrix} x\,s \\ y\,t \end{pmatrix}\right] = A'_1(s,t)H\begin{pmatrix} a\,t \\ y \end{pmatrix} + B'_1(s,t)H\begin{pmatrix} b\,t \\ y \end{pmatrix}.$$

The subscript 1 has been dropped from U_1 and V_1 for convenience, and the variable subscripts are inserted to indicate the variables operated on. On account of the relation (16) we have the identity

$$-U'_{xs}\left[V_{yt}\left[\bar{G}\begin{pmatrix} y\,t \\ x\,s \end{pmatrix}\right]\right] \equiv V_{yt}\left[U'_{xs}\left[G\begin{pmatrix} x\,s \\ y\,t \end{pmatrix}\right]\right],$$

which may be written

$$U'_{xs}\left[C_1(t,s)\bar{H}\begin{pmatrix} a\,s \\ x \end{pmatrix} + D_1(t,s)\bar{H}\begin{pmatrix} b\,s \\ x \end{pmatrix}\right]$$
$$+ V_{yt}\left[A'_1(s,t)H\begin{pmatrix} a\,t \\ y \end{pmatrix} + B'_1(s,t)H\begin{pmatrix} b\,t \\ y \end{pmatrix}\right] \equiv 0.$$

Expanding and collecting terms, we find

$$\left[\alpha'_1(s)\bar{H}\begin{pmatrix} a & s \\ a+ & \end{pmatrix} + \beta'_1(s)\bar{H}\begin{pmatrix} a\,s \\ b \end{pmatrix}\right]C_1(t,s) + \left[\alpha'_1(s)\bar{H}\begin{pmatrix} b\,s \\ a \end{pmatrix}\right.$$
$$\left. + \beta'_1(s)\bar{H}\begin{pmatrix} b & s \\ b- & \end{pmatrix}\right]D_1(t,s) + \left[\gamma_1(t)H\begin{pmatrix} a & t \\ a+ & \end{pmatrix}\right.$$
$$\left. + \delta_1(t)H\begin{pmatrix} a\,t \\ b \end{pmatrix}\right]A'_1(s,t) + \left[\gamma_1(t)H\begin{pmatrix} b\,t \\ a \end{pmatrix}\right.$$
$$\left. + \delta_1(t)H\begin{pmatrix} b & t \\ b- & \end{pmatrix}\right]B'_1(s,t) + \int_\alpha^\beta \left\{ A'_1(s,r)C_1(t,r)\left[\bar{H}\begin{pmatrix} a & r \\ a+ & \end{pmatrix}\right.\right.$$
$$\left.\left. + H\begin{pmatrix} a & r \\ a+ & \end{pmatrix}\right] + B'_1(s,r)D_1(t,r)\left[\bar{H}\begin{pmatrix} b & r \\ b- & \end{pmatrix} + H\begin{pmatrix} b & r \\ b- & \end{pmatrix}\right]\right\} dr$$
$$+ \int_\alpha^\beta \left\{ A'_1(s,r)D_1(t,r)\left[\bar{H}\begin{pmatrix} b\,r \\ a \end{pmatrix} + H\begin{pmatrix} a\,r \\ b \end{pmatrix}\right]\right.$$
$$\left. + B'_1(s,r)C_1(t,r)\left[\bar{H}\begin{pmatrix} a\,r \\ b \end{pmatrix} + H\begin{pmatrix} b\,r \\ a \end{pmatrix}\right]\right\} dr \equiv 0.$$

By means of the relations

$$H\begin{pmatrix} y\pm, s \\ y \end{pmatrix} = H\begin{pmatrix} y & s \\ y\mp & \end{pmatrix}$$

and (7), (16), (4), the first member of this equation reduces precisely to the expression $N_2(s, t)$. Thus our proof is completed.

COROLLARY. *A necessary and sufficient condition that the Green's functions of a system be skew-symmetric, i. e.,*

$$H\begin{pmatrix} x\,s \\ y \end{pmatrix} = -H\begin{pmatrix} y\,s \\ x \end{pmatrix}, \qquad G\begin{pmatrix} x\,s \\ y\,t \end{pmatrix} = -G\begin{pmatrix} y\,t \\ x\,s \end{pmatrix},$$

is that the integro-differential expression $L[u]$ be anti-self-adjoint and the boundary condition $U[u] = 0$ be self-adjoint.

The sufficiency of this theorem follows from Theorem IV, and the necessity from the theorem just proved.

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ON SCALAR AND VECTOR COVARIANTS OF LINEAR ALGEBRAS*

BY

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Introduction

1. Relation to the literature. This paper concerns itself with two rather different kinds of covariants of the general linear algebra, which might, for convenience, be distinguished by the adjectives scalar and vector. Consider the general linear algebra† E with the units $e_1, \cdots, e_n$ and with the constants of multiplication γ_{ijk} $(i, j, k = 1, \cdots, n)$, where the general number of the algebra is $X = \sum x_i e_i$. In a previous paper‡ we defined a rational integral covariant C of the algebra E as a rational integral function of the γ's and the x's which possesses the invariantive property whenever the units are subjected to a linear transformation. The paper just mentioned uses such covariants to characterize linear associative algebras in two and three units. The present paper proves for such covariants several fundamental theorems analogous to the fundamental theorems in the theory of invariants for algebraic forms. Since these covariants are isobaric and homogeneous, we can apply here Hilbert's proof of the " finiteness " of the number of covariants of algebraic forms.

We must not, however, content ourselves with the study of such covariants, since they are not sufficient to characterize linear algebras, both associative and non-associative, even when there are only two units. Accordingly, we consider rational integral functions of the γ's, the x's and also the units e_i $(i = 1, \cdots, n)$ which possess the invariantive property. Since these functions involve the e's, we call them *vector covariants* in contradistinction to the functions C mentioned above, which we might call *scalar covariants*. At first sight, it might seem as if we were confronted with a rather inconvenient difficulty for the following reason. In a theory of such covariants we would expect to find treated certain differential operators analogous to the familiar annihilators Ω and O; but Scheffers has shown that, in a linear algebra, a

* Part I was presented to the Society December 27, 1917, under a different title; Part II was presented February 23, 1918.

† For the fundamental definitions in the theory of linear algebras, see Section 2. The most familiar linear algebras are number fields (in particular, the field of reals and the field of ordinary complex numbers) and quaternions.

‡ Annals of Mathematics, second series, vol. 16 (1914), pp. 1–6.

derivative is not uniquely determined unless multiplication is commutative, and we are concerned with algebras both commutative and non-commutative. This Gordian knot can, however, be readily cut by a device. We can, accordingly, derive the annihilators in the approved manner, and hence show that every rational integral vector covariant of the linear algebra is a covariant of the general number of the algebra $X = \sum x_i e_i$ and a suitable set of scalar covariants of the algebra. From this fact flow theorems analogous to those proved for scalar covariants and, in particular, the "finiteness" of vector covariants.

2. Definitions. A *linear algebra* E over the field F is a set of hypercomplex numbers of the form $X = \sum_{i=1}^{n} x_i e_i$, where the *coördinates* x_i range independently over F. Here the *units* e_i are such that $e_i e_j = \sum_{k=1}^{n} \gamma_{ijk} e_k$ $(i, j = 1, \cdots, n)$, where (i) the *constants of multiplication* γ_{ijk} are in F, (ii) the sum of two numbers X and Y of the algebra is $X + Y = \sum_{i=1}^{n} (x_i + y_i) e_i$ and (iii) numbers of the algebra combine under addition and multiplication according to the distributive law. Unless explicitly stated, we do not assume the commutative nor the associative law of multiplication.*

If we have a number $X = \sum x_i e_i$ of an algebra whose coördinates are (ordinary) complex numbers, then we can find a number $Y = \sum y_i e_i \neq 0$ and a scalar ω such that $XY = \omega Y$ if and only if the *right-hand characteristic determinant* $\delta(\omega) \equiv |\sum_{i=1}^{n} \gamma_{ijk} x_i - d_{jk}\omega|$ is zero. Here $d_{jj} = 1$, $d_{jk} = 0$ if $k \neq j$. Similarly, there is a number $Y \neq 0$ and a scalar ω such that $YX = \omega Y$ if and only if the *left-hand* characteristic determinant

$$\delta'(\omega) \equiv |\textstyle\sum_{i=1}^{n} \gamma_{jik} x_i - d_{jk}\omega|$$

vanishes.

For convenience in the study of covariants of a linear algebra, we introduce the notion of *weight*. If we subject the algebra E to the transformation

$$(\lambda)_r \qquad \begin{aligned} e'_r &= \lambda e_r \qquad (\lambda \neq 0) \\ e'_i &= e_i \qquad (i \neq r), \end{aligned}$$

the γ's are subject to the induced transformation where γ_{ijk} is unaltered when $i, j, k \neq r$, γ_{ijr} $(i, j \neq r)$ is multiplied by λ^{-1}, γ_{rir} and γ_{irr} $(i \neq r)$ are unaltered, γ_{rrr} is multiplied by λ, γ_{rjk} and γ_{jrk} $(j, k \neq r)$ by λ and γ_{rrk} $(k \neq r)$ by λ^2. Hence we shall say that γ_{ijk} $(i, j, k \neq r)$ is of weight 0, γ_{ijr} $(i, j \neq r)$ of weight -1 in e_r, γ_{rir} and γ_{irr} $(i \neq r)$ of weight 0, γ_{rrr} of weight 1, γ_{rjk} and γ_{jrk} $(j, k \neq r)$ of weight 1 and γ_{rrk} $(k \neq r)$ of weight 2 in e_r. For every unit e_r we thus assign to γ_{ijk} a weight with respect to that unit, which is counted in the following way: the first subscript, i, contributes $+1$ if $i = r$, other-

* For definitions of a linear associative algebra by a set of independent postulates, see Dickson, these Transactions, vol. 4 (1903), pp. 21–26.

wise 0 ; the second, j, contributes $+1$ if $j = r$, otherwise 0 ; and the third, k, contributes -1 if $k = r$, otherwise 0. The sum of these three partial weights is called the weight of γ_{ijk} with respect to e_r.

For a similar reason, we shall say that e_r is of weight 1 in e_r but of weight 0 in any other unit e_i $(i \neq r)$.

If $X = \sum x_i e_i$ is the general number of the algebra E, then if $x'_1, \cdots, x'_n$ are the coördinates of X when it is expressed in terms of the new units—that is, $X = \sum x'_i e'_i$—then $x'_r = x_r/\lambda$ and $x'_i = x_i$ $(i \neq r)$. For this reason one might be tempted to say that x_r is of weight -1 with respect to e_r and that the other x's are of weight 0 with respect to e_r. In the applications, however, it will be better so to define weight that every x has a weight which is positive or zero with respect to any unit e_r. Accordingly, we shall agree that x_r is of weight 0 in e_r and that x_i $(i \neq r)$ is of weight 1 in e_r.

We shall use the term *total weight* of a term for the sum of the weights of that term in all the units $e_1, \cdots, e_n$. Notice that the total weight of any γ, x_i or e_i is 1, and thus the total weight of a product of a number of γ's, x's and e's is the degree of that term.

If P is the product of a number of γ's (or a number of e's) such that it is of weight w_r in a particular unit, e_r, it is multiplied by λ^{w_r} under the transformation $(\lambda)_r$. If, however, P is the product of a number of x's such that it is of degree s in the x's and of weight w_r in e_r, then under the transformation $(\lambda)_r$ it is multiplied by $\lambda^{w_r - s}$.

Any polynomial in the γ's, the x's and the e's such that all terms have the same weight with respect to a particular unit e_r will be said to be *isobaric with respect to* e_r. If the terms have the same total weight in all the units, the polynomial will be said to be *isobaric on the whole.* If P is a polynomial in the γ's, the x's and the e's which is isobaric with respect to e_r, of weight w_r in e_r and which is homogeneous in the x's of degree s, it is multiplied by $\lambda^{w_r - s}$ under the transformation $(\lambda)_r$.

Part I. Scalar Covariants

3. Some fundamental theorems. The invariancy of I under the transformation

$$T: \qquad e'_l = \sum_{m=1}^{n} a_{lm} e_m \qquad (l = 1, \cdots, n),$$

$$A = |a_{lm}| \neq 0,$$

is expressed by the formula

$$I(\gamma'_{ijk}) = \phi(a_{lm}) I(\gamma_{ijk}) \tag{1}$$

where the γ_{ijk} are the constants of multiplication of the original algebra and

the γ'_{ijk} are the constants of multiplication of the transformed algebra. Here $I(\gamma'_{ijk})$ has been written for $I(\gamma'_{111}, \gamma'_{112}, \cdots, \gamma'_{nnn})$, $\phi(a_{lm})$ for $\phi(a_{11}, a_{12}, \cdots, a_{nn})$ and $I(\gamma_{ijk})$ for $I(\gamma_{111}, \gamma_{112}, \cdots, \gamma_{nnn})$. Hence by applying the special transformation $e'_i = \lambda e_i$ ($\lambda \neq 0$; $i = 1, \cdots, n$), it follows that I is homogeneous in the γ's.

Under this transformation T the γ_{ijk} are subject to an induced transformation such that

$$(2) \qquad \sum_p \gamma'_{ijp}\, a_{pq} = \sum_{k,\,l} a_{ik}\, a_{jl}\, \gamma_{klq} \qquad (i, j, q = 1, \cdots, n).$$

If we keep i and j fixed in (2) and let q range from 1 to n, we have n non-homogeneous linear equations in the n unknowns γ'_{ijp} which can be solved uniquely, since $A \neq 0$. Hence each γ'_{ijp} is a homogeneous function of degree $n + 1$ in the a's divided by A.

Accordingly, if $I(\gamma_{ijk})$ is a rational integral invariant of total weight w in the units and hence homogeneous of degree w in the γ's, $I(\gamma'_{ijk})$ is a rational integral homogeneous function of degree $(n + 1)w$ in the a_{lm} divided by A^w. But, since $I(\gamma_{ijk})$ is independent of the a's, this implies that $\phi(a_{lm})$ is a rational integral homogeneous function $\psi(a_{lm})$ of degree $(n + 1)w$ in the a's divided by A^w. That is,

$$I(\gamma'_{ijk}) = \frac{\psi(a_{lm})}{A^w} I(\gamma_{ijk})$$

for every transformation of determinant $A \neq 0$. Hence under the inverse transformation T^{-1}

$$I(\gamma_{ijk}) = \frac{\psi\left(\dfrac{A_{ml}}{A}\right)}{(A^{-1})^w} I(\gamma_{ijk}'),$$

where A_{ml} is the cofactor of a_{ml}. Combining these two results, we see that

$$\psi(a_{lm})\,\psi\left(\frac{A_{ml}}{A}\right) = 1;$$

or, since ψ is homogeneous of degree $(n + 1)w$,

$$\psi(a_{lm})\,\psi(A_{ml}) = A^{(n+1)w}.$$

But, since A is irreducible, this can be true only if $\psi(a_{lm}) = kA^u$, where k is some constant. Then $\psi(A_{ml}) = kA'^u$, where $A' = |A_{ml}| = A^{n-1}$, and hence $u = w/n$. By applying the identical transformation, it is evident that $k = 1$, and thus

$$(3) \qquad \phi(a_{lm}) = A^{w/n}.$$

This last formula can be proved in the above manner familiar in the classical invariant theory, or it can also be proved for associative algebras by the

following device. If $X = \sum x_i e_i$ is a number of the general n-ary linear algebra $E = (e_1, \cdots, e_n)$, then $EX = E = XE$. That is, if we multiply each unit e_i of E by X, we obtain n linearly independent numbers of E which can be taken as a new set of units. This remark still applies if we restrict ourselves to the general associative algebra. Accordingly, replace e_i by Xe_i $(i = 1, \cdots, n)$. Then

$$e_i' e_j' \equiv (Xe_i)(Xe_j) = \sum_m \Big(\sum_{k,l} x_k \gamma_{kjl} \gamma_{ilm} \Big) (Xe_m) \qquad (i, j = 1, \cdots, n).$$

Thus $I(\gamma'_{ijk})$ is of degree $2w$ in the γ's. But A^r is of degree rn and $I(\gamma_{ijk})$ is of degree w in these γ's. Hence $2w = rn + w$, or $r = w/n$.

Also, if w_i is the weight of I in e_i, then $w_1 = w_2 = \cdots = w_n$. This follows from equations (1) and (3) and the fact that I must be unaltered (except possibly for sign) when any two units, say e_i and e_j, are interchanged.

Since w/n is the weight of I in any particular unit, say e_1, w/n is a positive integer. Notice, moreover, that w is equal to the degree of I, since each γ_{ijk} is of total weight 1 in the units. Thus we have

Theorem I. *For an n-ary linear algebra $E = (e_1, p \cdots, e_n)$ with constants of multiplication γ_{ijk}, let I be a rational integral invariant of degree d. Then I is homogeneous and isobaric of total weight d in all the units, and d/n is a positive integer. If we subject the units of E to the transformation T of determinant $A \neq 0$, carrying E into the algebra E' with constants of multiplication γ'_{ijk}, then*

$$I(\gamma'_{ijk}) = A^{d/n} I(\gamma_{ijk}).$$

The covariancy of the rational integral function C under the transformation T is expressed by the formula

$$C(\gamma'_{ijk};\, x'_l) = \phi(a_{lm}) C(\gamma_{ijk};\, x_l).$$

Such a covariant is not necessarily homogeneous in the coördinates x_i of the general number of the algebra, but it is the sum of covariants C_k such that each C_k is homogeneous in the x's. Accordingly, we may, without loss of generality, restrict ourselves to the study of homogeneous covariants. Every covariant C which is homogeneous in the x's is also homogeneous in the γ's. If C is homogeneous of degree s in the x's and of degree d in the γ_{ijk}, then $C(\gamma'_{ijk};\, x'_l)$ is a rational integral homogeneous function of degree $(n+1)d + (n-1)s$ in the a's, divided by A^{d+s}. Thus $\phi(a_{lm})$ in this case is a rational integral homogeneous function, ψ, of the a's of degree $(n+1)d + (n-1)s$ divided by A^{d+s}, and consequently

$$\psi(A_{ml}) \psi(a_{lm}) = A^{(n+1)d+(n-1)s}.$$

Thence it follows that

$$\phi(a_{lm}) = A^{(d-s)/n}.$$

When we subject E to the special transformation

$$e'_i = e_i\,(i \neq n), \qquad e'_n = \lambda e_n\,(\lambda \neq 0),$$

each term of $C(\gamma'_{ijk})$ is equal to the corresponding term of $C(\gamma_{ijk})$ multiplied by $\lambda^{w_n - s}$, where w_n is the weight of that term in e_n. This follows at once from the remarks at the end of section 2, since this is the transformation $(\lambda)_n$. But under this transformation, $C(\gamma_{ijk})$ is multiplied by $\lambda^{(d-s)/n}$ and thus $w_n - s = (d-s)/n$ or $w_n = [d + (n-1)s]/n$. It follows that the index of the power of A—namely $(d-s)/n$—is an integer. Moreover, it is not negative, for it is the weight of the coefficient of a term of C which is independent of x_n. Now such a term actually occurs in C, as otherwise x_n would be a factor of C, and hence (in view of the covariancy of C) every linear function of the x's would be a factor of C, which is impossible. Thus we have proved

THEOREM II. *For an n-ary linear algebra $E = (e_1, \cdots, e_n)$ where the constants of multiplication are γ_{ijk} and the general number of the algebra is $X = \sum x_i e_i$, let C be a rational integral covariant of degree d in the γ_{ijk} and degree s in the x_i. Then C is isobaric of total weight $d + (n-1)s$ in all the units. If C is homogeneous in the x's, it is homogeneous in the γ's. Moreover, if we subject the units of E to the transformation T of determinant $A \neq 0$, carrying E into the algebra E′ with constants of multiplication γ'_{ijk}, then*

$$C(\gamma'_{ijk};\, x'_l) = A^{(d-s)/n}\, C(\gamma_{ijk};\, x_l).$$

*The index, $(d-s)/n$, is an integer, positive or zero.**

4. Fundamental sets of invariants and covariants for $n = 1, 2$. When $n = 1$, the multiplication table is $e_1 e_1 = \gamma e_1$ and the general number of the algebra is $x_1 e_1$. Hence $\delta(\omega) = \delta'(\omega) = \omega - \gamma x_1$; and the only rational integral invariants in this case are powers of γ, and the only rational integral scalar covariants of the unary algebra are all polynomials in γ and x_1.

When $n = 2$, the multiplication table is of the form $e_i e_j = \gamma_{ij1} e_1 + \gamma_{ij2} e_2$ $(i, j = 1, 2)$ and the general number of the algebra is $X = x_1 e_1 + x_2 e_2$. Hence

$$\delta(\omega) = \omega^2 - \omega[\Gamma_1 x_1 + \Gamma_2 x_2] + [\Gamma_5 x_1^2 + \Gamma_9 x_1 x_2 + \Gamma_6 x_2^2],$$

$$\delta'(\omega) = \omega^2 - \omega[\Gamma_3 x_1 + \Gamma_4 x_2] + \Gamma_7 x_1^2 + \Gamma_{10} x_1 x_2 + \Gamma_8 x_2^2],$$

where

$$\Gamma_1 = \gamma_{111} + \gamma_{122}, \qquad \Gamma_2 = \gamma_{211} + \gamma_{222},$$

$$\Gamma_3 = \gamma_{111} + \gamma_{212}, \qquad \Gamma_4 = \gamma_{121} + \gamma_{222},$$

* By the use of this theorem, we can readily prove the following corollary: Every rational integral invariant and covariant of the general linear algebra vanishes for a nilpotent algebra. This theorem was stated and proved for invariants in a paper in the Annals of Mathematics, second series, vol. 18 (1916), p. 84, and the proof for covariants proceeds in essentially the same way.

$$\Gamma_5 = \gamma_{111}\gamma_{122} - \gamma_{112}\gamma_{121}, \qquad \Gamma_6 = \gamma_{211}\gamma_{222} - \gamma_{212}\gamma_{221},$$

$$\Gamma_7 = \gamma_{111}\gamma_{212} - \gamma_{112}\gamma_{211}, \qquad \Gamma_8 = \gamma_{121}\gamma_{222} - \gamma_{122}\gamma_{221},$$

$$\Gamma_9 = \gamma_{111}\gamma_{222} + \gamma_{211}\gamma_{122} - \gamma_{121}\gamma_{212} - \gamma_{112}\gamma_{221},$$

$$\Gamma_{10} = \gamma_{111}\gamma_{222} + \gamma_{121}\gamma_{212} - \gamma_{122}\gamma_{211} - \gamma_{112}\gamma_{221}.$$

We may prove without any difficulty that, for a binary algebra, every homogeneous isobaric function of the constants of multiplication for which $w_1 = w_2$ and which is annihilated by

$$U_{12} = (\gamma_{121} + \gamma_{211})\frac{\partial}{\partial\gamma_{111}} + (\gamma_{222} - \gamma_{121})\frac{\partial}{\partial\gamma_{122}} + (\gamma_{222} - \gamma_{211})\frac{\partial}{\partial\gamma_{212}}$$
$$+ [(\gamma_{212} + \gamma_{122}) - \gamma_{111}]\frac{\partial}{\partial\gamma_{112}} + \gamma_{221}\left[\frac{\partial}{\partial\gamma_{121}} + \frac{\partial}{\partial\gamma_{211}} - \frac{\partial}{\partial\gamma_{222}}\right]$$

is an invariant of the algebra. We can prove a similar theorem for scalar covariants. Accordingly, one may readily determine the rational integral scalar covariants of a given weight, and so one might hope to find a fundamental set of covariants for the binary algebra.

One can, however, avoid such troubles in view of

THEOREM III. *Every scalar covariant of the binary algebra $E = (e_1, e_2)$ where the constants of multiplication are γ_{ijk} and where the general number of the algebra is $X = x_1 e_1 + x_2 e_2$ is a covariant of the characteristic determinants δ and δ'.*

We want to show that any rational integral function $C(\gamma_{ijk}; x_l)$ of the eight γ's and the x's which has the invariantive property under the group of linear transformations on e_1 and e_2 is a function of $\Gamma_1, \cdots, \Gamma_8$ and the x's which is a covariant of the system of forms $\Gamma_1 x_1 + \Gamma_2 x_2$, $\Gamma_3 x_1 + \Gamma_4 x_2$, $\Gamma_5 x_1^2 + \Gamma_9 x_1 x_2 + \Gamma_6 x_2^2$, $\Gamma_7 x_1^2 + \Gamma_{10} x_1 x_2 + \Gamma_8 x_2^2$ under the group of linear transformations on x_1 and x_2. Since x_1 and x_2 are contragredient to e_1 and e_2, $C(\gamma_{ijk}; x_l)$ has the invariantive property under the group of linear transformations on x_1 and x_2 where the γ's are transformed according to (2). Thus we can prove our theorem if we can show that any function of the γ's is a function of $\Gamma_1, \cdots, \Gamma_8$. Now this follows from the fact that the Jacobian of $\Gamma_1, \cdots, \Gamma_8$ is

$$J = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \gamma_{122} & 0 & -\gamma_{121} & 0 & -\gamma_{112} & 0 & 0 & \gamma_{111} \\ 0 & \gamma_{211} & 0 & -\gamma_{212} & 0 & -\gamma_{221} & \gamma_{222} & 0 \\ \gamma_{212} & 0 & -\gamma_{211} & 0 & 0 & \gamma_{111} & -\gamma_{112} & 0 \\ 0 & \gamma_{121} & 0 & -\gamma_{122} & \gamma_{222} & 0 & 0 & -\gamma_{221} \end{vmatrix};$$

which is not identically zero, since the terms free of γ_{112}, γ_{111}, γ_{222} and γ_{221} equal the four-rowed determinant in the lower left-hand corner.

5. Finiteness of the rational, integral scalar covariants. In view of the homogeneity and isobarism of rational integral scalar covariants, we can readily prove that they are all expressible as polynomials in a finite subset. For, by Hilbert's theorem about an infinite sequence of polynomials,* any such covariant C can be expressed in the form

$$C(\gamma;\ x) = \sum K_j(\gamma;\ x)C_j(\gamma;\ x) \tag{5}$$

where the C_j are scalar covariants of the algebra which are the same for every C. Since C and C_j are homogeneous and isobaric, we may suppose that the K_j are also. If we subject E to the transformation T of determinant $A \neq 0$, (5) is replaced by

$$A^w C(\gamma;\ x) = \sum A^{w_j} K_j(\gamma';\ x')C_j(\gamma;\ x), \tag{6}$$

where C is of index w and C_j is of index w_j. Now operate on both sides of (6) with

$$\Omega_a \equiv \sum (\pm i_1 \cdots i_n) \frac{\partial^n}{\partial a_{1i_1} \cdots \partial a_{ni_n}}$$

until both sides of the resulting equation no longer involve the a's. Then, by a well-known theorem of Hilbert's (which is essentially one given by Mertens and Gordan†), (6) becomes

$$NC(\gamma;\ x) = \sum N_j C'_j(\gamma;\ x)C_j(\gamma;\ x),$$

where the C'_j are scalar covariants of the algebra, and N and N_j are constants different from zero. Thus we prove by induction

THEOREM IV. *For the general n-ary linear algebra there is a finite set* $C_1, \cdots, C_m$ *of scalar covariants such that every rational integral scalar covariant of the algebra is a polynomial in* $C_1, \cdots, C_m$.

COROLLARY. For the general n-ary linear algebra there is a finite set $B_1, \cdots, B_p$ of rational scalar covariants such that every rational integral scalar covariant of the algebra is a covariant of this set of algebraic forms. For convenience, such a *set of covariants* will be called *basic*.

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* "*Ueber die Theorie der algebraischen Formen*," Mathematische Annalen, vol. 36 (1890), p. 474.

† Hilbert, loc. cit., p. 524; Gordan, *Beweis, dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist*, Journal für Mathematik, vol. 69 (1868), pp. 323–354; *Die simultanen Systeme binärer Formen*, Mathematische Annalen, vol. 2 (1870), pp. 227–280; *Invariantentheorie* (1887), vol. II, § 9; Mertens, *Beweis dass alle Invarianten und Covarianten* . . ., Journal für Mathematik, vol. 100 (1887), pp. 223–230.

Part II. Vector Covariants

6. Preliminary remarks. As we pointed out in the introduction, scalar covariants are not sufficient to characterize linear algebras—in fact, they are not sufficient to characterize potent algebras in 2 units, for the two algebras

$$e_1 e_1 = e_1, \quad e_1 e_2 = e_2 e_1 = 0, \quad e_2 e_2 = \gamma e_1 \quad (\gamma > 0);$$

$$e_1 e_1 = e_1, \quad e_1 e_2 = e_2 e_1 = 0, \quad e_2 e_2 = 0$$

are not equivalent with respect to the field of reals, and yet for both algebras $\delta = \delta' = \omega(\omega - x_1)$ and both have the same rank and the same index.

Accordingly we shall now consider vector covariants. By a rational integral function of the γ's, the x's and the e's we shall understand any linear combination of a finite number of products of units of the algebra where the coefficients are polynomials in the γ's and x's. It must be borne in mind that, since multiplication is not in general commutative nor associative, we cannot, for instance, combine into one term two such as $e_1[(e_1 e_2)e_3]$ and $(e_1 e_1)(e_2 e_3)$. That is, in some terms the multiplication is to be carried out in one order and in other terms in a different order. Every rational integral vector covariant V is, however, the sum of a finite number of vector covariants V_k such that, in any one V_k, the multiplication indicated in every term is to be carried out in the same order.

7. Fundamental properties. We can readily show that if a vector covariant is homogeneous in the units, e_i, and homogeneous in the x_i, it is homogeneous in the γ_{ijk}. Moreover it is isobaric in any unit e_i, and its weight in e_j is equal to its weight in e_i $(i, j = 1, \cdots, n)$. From these properties we can readily prove the analogue of Theorem II, which we state here merely for the sake of completeness.

Theorem V. *For an n-ary linear algebra $E = (e_1, \cdots, e_n)$ where the constants of multiplication are γ_{ijk} and the general number of the algebra is $X = \sum x_i e_i$, let V be a rational integral vector covariant of degree d in the γ_{ijk}, of degree s in the x's and of degree v in the e's. Then V is isobaric of weight $w = d + (n-1)s + v$ in the units altogether. If V is homogeneous in the x's and the e's, it is homogeneous in the γ's. Furthermore, if we subject the units of E to the transformation T of determinant $A \neq 0$, carrying E into the algebra $E' = (e'_1, \cdots, e'_n)$ where the general number of the algebra is $X' = \sum x'_i e'_i$ and the constants of multiplication are γ'_{ijk}, then*

$$V(\gamma'_{ijk};\ x'_l;\ e'_m) = A^{(d-s+v)/n}\, V(\gamma_{ijk};\ x_l;\ e_m).$$

Moreover the index $(d - s + v)/n$ is an integer, positive or zero.

8. Annihilators for vector covariants. Since we are dealing with a *linear* algebra, every vector covariant whose degree in the units is greater than 2

can be made linear in the units. Henceforth, unless otherwise specified, we shall think of every vector covariant as linear in the units.

First consider absolute vector covariants of degree 1 in the x's. Now such a covariant is of the form

$$\sum_{i=1}^{n} [x_1 \phi_{1i}(\gamma) + x_2 \phi_{2i}(\gamma) + \cdots + x_n \phi_{ni}(\gamma)] e_i \tag{7}$$

where each ϕ_{ii} is a polynomial of weight zero in each unit, and each ϕ_{ij} $(i \neq j)$ is a polynomial of weight 1 in e_i, of weight -1 in e_j, and of weight zero in the other units. But any polynomial in the γ's of degree d is of total weight d in the e's altogether. Hence each $\phi_{ij} = 0$ $(i \neq j)$ and each ϕ_{ii} is a constant, c_i. But, since (7) is unaltered if we interchange any two units, the c_i are all equal, and thus a vector covariant which is of the first degree in the x's is necessarily a constant multiple of the general number of the algebra.

Next, consider absolute vector covariants of degree 2 in the x's; such a covariant is necessarily of the form

$$\sum_{i=1}^{n} [x_i^2 \phi_{iii}(\gamma) + x_i \sum_{j \neq i} x_j \phi_{iji}(\gamma) + \sum_{j,\, k \neq i} x_j x_k \phi_{jki}(\gamma)] e_i \tag{8}$$

where each ϕ_{iii} is of weight 1 in e_i and of weight zero in e_l $(l \neq i)$, each ϕ_{iji} $(i \neq j)$ is of weight 1 in e_j and of weight zero in e_l $(l \neq j)$, and each ϕ_{jki} $(j, k \neq i)$ is of weight -1 in e_i, of weight 1 in e_j and in e_k and of weight zero in e_l $(l \neq i, j, k)$. Accordingly each ϕ is of degree 1 in the γ's. In particular,

$$\phi_{111} = \alpha\gamma_{111} + \beta(\gamma_{122} + \gamma_{133} + \gamma_{1nn}) + \gamma(\gamma_{212} + \cdots + \gamma_{n1n}),$$

$$\phi_{121} = \delta\gamma_{121} + \epsilon(\gamma_{323} + \cdots + \gamma_{n2n}) + \zeta\gamma_{211} + \eta(\gamma_{233} + \cdots + \gamma_{2nn}) + \theta\gamma_{222},$$

$$\phi_{231} = \kappa\gamma_{231} + \lambda\gamma_{321}, \qquad \phi_{221} = \mu\gamma_{221},$$

and any ϕ_{iii} is obtained from ϕ_{111} by interchanging the subscripts 1 and i; any ϕ_{iji} $(j \neq i)$ is obtained from ϕ_{121} by interchanging 1 and i, and 2 and j; and so forth. Hence the difference between (8) and $\beta C_{n-1} X + \gamma C'_{n-1} X + (\alpha - \beta - \gamma) X^2$ is a covariant of the type (8) where the α, β, γ are zero. Here C_{n-1} and C'_{n-1} are the coefficients of ω^{n-1} in $\delta(\omega)$ and $\delta'(\omega)$ respectively. Hence consider the covariant (8) where α, β, γ are all zero. Since such a covariant must be unaltered under the transformation

$$e'_1 = e_1 + e_2, \qquad e'_i = e_i \ (i \neq 1),$$

we must have $\delta = \epsilon = \zeta = \eta = \theta = \kappa = \lambda = \mu = 0$. Therefore every ra-

tional integral absolute vector covariant of degree 2 in the x's is a homogeneous polynomial of degree 2 in C_{n-1}, C'_{n-1} and $X = \sum x_i e_i$.

More generally we have

THEOREM VI. *Every rational integral vector covariant of the n-ary linear algebra is a covariant of the general number of the algebra and any basic set of scalar covariants of the algebra.*

We prove this by using a set of annihilators for vector covariants—differential operators analogous to the familiar Aronhold operators Ω and O. In deriving such operators for the scalar covariants, we used Taylor's theorem with a remainder for polynomials in the γ's and the x's, and so we now seem to be confronted with the necessity of inquiring if this theorem (or one closely analogous) holds for polynomials in the γ's, the x's and the e's—which we might, for brevity, call vector polynomials. This brings forward the question of a differential calculus for such *vector* polynomials.

Now Scheffers* has considered the theory of functions of hypercomplex variables. In a linear algebra over the field C of ordinary complex numbers in which division is generally possible, he considers functions of the form $f = \sum f_i e_i$ where each f_i is an analytic function of the x_j $(j = 1, \cdots, n)$. When

$$dx = \sum_i dx_i e_i,$$

$$df = \sum_{i,j} \frac{\partial f_i}{\partial x_k} dx_k e_i.$$

Then Scheffers shows that such a linear algebra which possesses a principal unit permits of functions f with derivatives with respect to x which are independent of $\Delta x_1, \cdots, \Delta x_n$—that is, independent of the " direction " in which Δx approaches zero—if and only if multiplication is commutative.

It would seem, then, as if we could not hope to find annihilators for vector covariants of the *general* n-ary linear algebra. There are, however, several ways of avoiding this little difficulty.

As we pointed out at the beginning of this sect'on, we can think of a vector covariant as linear in the e's ; and so, to derive the desired annihilators, we need only to consider such derivatives as

$$\frac{\partial e_1}{\partial e_1}, \frac{\partial e_2}{\partial e_1}, \cdots.$$

With this end in view, let us consider the linear function $\phi(e_1) = e_1$. Re-

* *Verallgemeinerung der Grundlagen der gewöhnlichen complexen Functionen*, Leipziger Berichte, vol. 45 (1893), pp. 828–848; vol. 46 (1894), pp. 120–134. There are abstracts of these articles in Paris Comptes Rendus, vol. 116 (1893), pp. 1114–1117, 1242–1244.

placing e_1 by $e_1 + \Delta e_1$, where the increment Δe_1 is a scalar multiple of e_2, say me_2, we have $\Delta\phi = \Delta e_1$. Now divide on the right by Δe_1, i. e. find a number u such that $u \cdot \Delta e_1 = \Delta e_1$. If we are dealing with any *particular* algebra, this number u depends on *that* algebra; but we are interested in the *general* algebra, and thus we want a number u such that $u \cdot \Delta e_1 = \Delta e_1$ (or $u \cdot e_2 = e_2$) for *every* n-ary linear algebra. One such number u is the scalar 1. If we take $u = 1$, $de_1/de_1 = 1$. With a similar understanding,

$$\frac{\partial e_2}{\partial e_1} = 0,$$

$$\frac{\partial e_3}{\partial e_1} = 0, \qquad \cdots.$$

Similarly for the derivatives with respect to the other units.

The advantage of defining these derivatives in this way is that we can differentiate with respect to any unit, such as e_1, in a purely *formal* manner—precisely as if we were differentiating with respect to a scalar. Thus a vector covariant which is linear in the units is annihilated by the differential operators

$$U_{ij} + \sum \left(\frac{\partial x_k'}{\partial a_{ij}} \right) \cdot \frac{\partial}{\partial x_k} + \sum \left(\frac{\partial e_k'}{\partial a_{ij}} \right) \cdot \frac{\partial}{\partial e_k} \qquad (i, j = 1, \cdots, n),$$

where the U_{ij} are the annihilators for the invariants of the n-ary algebra.

Or we may reason as follows.

Every vector covariant when linear in the units has these annihilators, since such a covariant has the invariantive property when the e's are replaced by a set of cogredient scalars.

Theorem VI now follows at once from the fact that the annihilators for the vector covariants of the algebra are the same as those for covariants of a basic set of scalar covariants of the algebra and the general number of the algebra, $\sum x_i e_i$.

9. Finiteness of vector covariants. We now readily prove

THEOREM VII. *For the general n-ary linear algebra there is a finite set V_1, $\cdots$, V_m of rational integral vector covariants such that every vector covariant of the algebra is a linear function of $V_1, \cdots, V_m$.*

For every vector covariant of a linear algebra may be made linear in the e's. Now thinking of the e's as replaced by a set of cogredient scalars, which we shall call the y's, we have a set of rational integral functions of the variables x_i $(i = 1, \cdots, n)$ and the contragredient variables y_i $(i = 1, \cdots, n)$ together with the n^3 parameters γ_{ijk} $(i_{ij}, k = 1, \cdots, n)$ which are homogeneous and isobaric and which have the invariantive property under the total linear

group on the n variables x_i. Accordingly Hilbert's "finiteness" proof applies. Or this theorem can be proved as a corollary of Theorem VI.

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